EET 3200: Electric Circuit Theory /

EMT 3204: Principles of Electrical Technology

Contact Hours 45

Purpose of the Course

The aim of this course is to enable the student to understand basic circuit parameters and laws and their applications in electrical circuits.

Expected Learning Outcomes:

By the end of this course, the student should be able to;

1. Understand basic laws associated with DC circuits.
2. Understand magnetic circuits
3. Understand the difference between DC and AC circuits.
4. Define various electrical circuit parameters
5. Analyse sine waves and three-phase circuits
6. Understand AC circuit analysis.
7. Understand RLC circuits.

Course Content

Electrical power sources: Electric power, Joule's law, Resistance, current and voltage. **Circuits and networks**: Passive and active components; capacitors, resistors, inductors. **Series and parallel circuits:** Ohm's, and Kirchhoff's laws, Direct Current (D.C.) circuit analysis, Electric fields and capacitance, Magnetic circuits, Self-inductance, Mutual inductance. Transient analysis of D.C. circuits with resistance, inductance and capacitance (RLC), Step response, rise time, impulse response. **Alternating voltage and current**: Sine waves, mean, peak and root mean square (r.m.s.) values. Impedance and admittance. Power in Alternating Current (A.C.) circuits, Analysis of single phase A.C. circuits, power factor, and three-phase A.C.: Analysis of three-phase A.C. circuits. Phasors.

Mode of Delivery:

- Lecture
- Class discussions,
- Demonstrations
- Multimedia (computer, Internet, CD, DVD, film)
- Laboratory/workshop practicals
- assignments

Course Assessment:

Type	Weighting (%)
Examination	60
Continuous Assessment	10
Practical	25
Assignment	5
Total	100

Core Reading Material:

1. Hughes E. (2002) Electrical and Electronic Technology, Prentice Hall.
2. Theraja, B.L. &Theraja, A.K. (1997) Electrical technology, Vol. I, Nirja Construction and Development Company Pvt. Ltd. New Delhi 22nd Ed.
3. Gupta J.B. (2009): Fundamentals of Electrical Engineering and Electronics .S.K. Kataria & Sons 8th edition.
4. Warne D.F. (2005): Newness electrical power engineering handbook. Elsevier 2nd edition
5. Collins R. Bayliss (1999): Transmission and Distribution Electrical Engineering 4th edition

1 International System of Measurement

The International System of Units, known as **SI** in every language, was formally introduced in **1960** and has been accepted by most countries as their only legal system of measurement.

1.1 Basic SI Units

The SI is based on the measures of **six physical quantities**:

i). Mass.	Symbol: **m**	Unit: **kilogram (kg)**
ii). Length.	Symbol: **l**	Unit: **metre (m)**
iii). Time	Symbol: **t**	Unit: **second (s)**
iv). Electric current.	Symbol: **I**	Unit: **ampere (A)**

v). Absolute temperature (kelvin, K)

vi). Luminous intensity (candela, cd)

1.2 Derived SI units

All other units are derived units and are related to these base units by definition. Examples are:

Area	Symbol: A	Unit: square metre (m^2)
Volume	Symbol: V	Unit: cubic metre (m^3)
Velocity	Symbol: u	Unit: metre per second (m/s)
Acceleration	Symbol: a	Unit: metre per second squared (m/s^2)
Angular velocity	Symbol: ω	Unit: radian per second (rad/s)
Force	Symbol: F	Unit: newton (N)

$$F = ma \qquad [1.1]$$

F [newtons] = m [kilograms] × a [metres per second^2]

1.2.1 Unit of turning moment or torque

If a force F, in Newtons, is acting at right angles to a radius r, in metres, from a point, the turning moment or torque about that point is

Fr newton metres

Torque Symbol: T (or M) Unit: **newton metre (N m)**

If the perpendicular distance from the line of action to the axis of rotation is r, then

$$T = Fr \qquad [1.3]$$

The symbol M is reserved for the torque of a rotating electrical machine.

1.2.2 Unit of work or energy

Work or energy Symbol: W Unit: **joule (J)**

$$W = Fl \qquad [1.4]$$

The joule is the work done when a force of 1N acts through a distance of 1m in the direction of the force. Hence, if **a force F acts through distance l in its own direction**:

$$\text{Work done} = F\,[\text{newtons}] \times l\,[\text{metres}]$$
$$= Fl \text{ joules}$$

Note that energy is the capacity for doing work. Both energy and work are therefore measured in similar terms.

If a body having mass m, in kilograms, is moving with velocity u, in metres per second.

$$\text{Kinetic energy} = \tfrac{1}{2}mu^2 \text{ joules}$$

$$\therefore \quad W = \tfrac{1}{2}mu^2$$

If a body having mass m, in kilograms, is lifted vertically through height h, in metres, and if g is the gravitational acceleration, in metres per second squared, in that region, the potential energy acquired by the body is

Work done in lifting the body = mgh joules

$$W \simeq 9.81mh \qquad [1.6]$$

Example 1.3 A body having a mass of 30 kg is supported 50 m above the earth's surface. What is its potential energy relative to the ground?

If the body is allowed to fall freely, calculate its kinetic energy just before it touches the ground. Assume gravitational acceleration to be 9.81 m/s^2.

Weight of body = 30 [kg] × 9.81 [N/kg] = 294.3 N

$\therefore$ Potential energy = 294.3 [N] × 50 [m] = **14 700 J**

Note: here we carried a false accuracy in the figure for the weight and rounded the final answer to three figures.

If u is the velocity of the body after it has fallen a distance l with an acceleration g

$$u = \sqrt{(2gl)} = \sqrt{(2 \times 9.81 \times 50)} = 31.32 \text{ m/s}$$

and

$$\text{Kinetic energy} = \tfrac{1}{2} \times 30 \text{ [kg]} \times (31.32)^2 \text{ [m/s]}^2 = \mathbf{14\ 700\ J}$$

Hence the whole of the initial potential energy has been converted into kinetic energy. When the body is finally brought to rest by impact with the ground, practically the whole of this kinetic energy is converted into heat.

1.2.3 Unit of power

Since **power is the rate of doing work**, it follows that the SI unit of power is the ***joule per second*, or *watt.***

Power Symbol: P Unit: **watt (W)**

$$P = \frac{W}{t} = \frac{F \cdot l}{t} = F \cdot \frac{l}{t}$$

$$P = Fu \qquad [1.7]$$

In the case of a rotating electrical machine:

$$P = M\omega = \frac{2\pi N_r M}{60} \qquad [1.8]$$

where N_r is measured in revolutions per minute.

Rotational speed Symbol: N_r Unit: **revolution per minute (r/min)**

In the SI, the rotational speed ought to be given in revolutions per second but this often leads to rather small numbers, hence it is convenient to give rotational speed in revolutions per minute. The old abbreviation was rev/min and this is still found to be widely in use.

Rotational speed Symbol: n_r Unit: **revolution per second (r/s)**

1.2.4 Unit of Electric Charge

The **unit of charge** is the coulomb (C) where one coulomb is one ampere second. The coulomb is defined as the quantity of electricity which flows past a given point in an electric circuit when a current of one ampere is maintained for one second.

$$\boldsymbol{Q = It}$$

where I is the current in amperes and t is the time in seconds.

Problem 1. If a current of 5 A flows for 2 minutes, find the quantity of electricity transferred.

Quantity of electricity $Q = It$ coulombs

$I = 5$ A, $t = 2 \times 60 = 120$ s

Hence $Q = 5 \times 120 = \mathbf{600\ C}$

1.2.5 Unit of Electrical potential and e.m.f.

The **unit of electric potential** is the volt (V) where one volt is one joule per coulomb. One volt is defined as the difference in potential between two points in a conductor which, when carrying a current of one ampere, dissipates a power of one watt.

A change in electric potential between two points in an electric circuit is called a **potential difference**. The **electromotive force (e.m.f.)** provided by a source of energy such as a battery or a generator is measured in volts.

1.2.6 Unit of Resistance and conductance

The **unit of electric resistance** is the **ohm (Ω)** where one ohm is one volt per ampere.

$$R = \frac{V}{I}$$

where V is the potential difference across the two points in volts and I is the current flowing between the two points in amperes.

The reciprocal of resistance is called **conductance** and is measured in siemens (S). Thus,

conductance, in siemens $G = \frac{1}{R}$

where R is the resistance in ohms.

1.2.7 Electrical power and energy

When a direct current of I amperes is flowing in an electric circuit and the voltage across the circuit is V volts, then

power, in watts $\boxed{P = VI}$

Electrical energy = Power × time

$= VIt$ **Joules**

There is another unit of energy which is used commercially: the **kilowatt hour (kWh).** It represents the work done by working at the rate of one kilowatt for a period of one hour. Once known as the Board of Trade Unit, it is still widely referred to, especially by electricity suppliers, as the unit.

1 kW h = 1000 watt hours

= 1000 × 3600 watt seconds or joules

= 3 600 000 J = 3.6 MJ

Problem 7. A source e.m.f. of 5 V supplies a current of 3 A for 10 minutes. How much energy is provided in this time?

Energy = power × time and power = voltage × current. Hence

Energy $= VIt = 5 \times 3 \times (10 \times 60) = 9000$ Ws or J

$= \mathbf{9\ kJ}$

Problem 8. An electric heater consumes 1.8 MJ when connected to a 250 V supply for 30 minutes. Find the power rating of the heater and the current taken from the supply.

i.e. **Power rating of heater = 1 kW**

Power $P = VI$, thus $I = \dfrac{P}{V} = \dfrac{1000}{250} = 4$ A

Hence the current taken from the supply is 4 A

Example 1.5 An electric motor is developing 10 kW at a speed of 900 r/min. Calculate the torque available at the shaft.

$$\text{Speed} = \frac{900\ [\text{r/min}]}{60\ [\text{s/min}]} = 15\ \text{r/s}$$

Substituting in expression [1.8], we have

$$10\,000\ [\text{W}] = T \times 2\pi \times 15\ [\text{r/s}]$$

$$\therefore \quad T = \mathbf{106\ N\ m}$$

1.3 Temperature

Some mention is required of temperature measurement, which is in the Celsius scale. Absolute temperature is measured in kelvin, but for most electrical purposes at an introductory stage it is sufficient to measure temperature in degrees Celsius.

It should be remembered that both degrees of temperature represent the same change in temperature – the difference lies in the reference zero.

Temperature Symbol: θ Unit: **degree Celsius (°C)**

A useful constant to note is that it takes 4185 J to raise the temperature of 1 litre of water through 1 °C.

Example 1.8 An electric heater contains 40 litres of water initially at a mean temperature of 15 °C; 2.5 kW h is supplied to the water by the heater. Assuming no heat losses, what is the final mean temperature of the water?

$$W_{in} = 2.5 \times 3.6 \times 10^6 = 9 \times 10^6 \text{ J}$$

Energy to raise temperature of 40 litres of water through 1 °C is

$$40 \times 4185 \text{ J}$$

Therefore change in temperature is

$$\Delta\theta = \frac{9 \times 10^6}{40 \times 4185} = 53.8 \text{ °C}$$

$$\theta_2 = \theta_1 + \Delta\theta = 15 + 53.8 = 68.8 \text{ °C}$$

SI units may be made larger or smaller by using prefixes which denote multiplication or division by a particular amount. The **six most common multiples**, with their meaning, are listed in the Table below:

Prefix	*Name*	*Meaning*	
M	mega	multiply by 1 000 000	(i.e. $\times 10^6$)
k	kilo	multiply by 1000	(i.e. $\times 10^3$)
m	milli	divide by 1000	(i.e. $\times 10^{-3}$)
μ	micro	divide by 1 000 000	(i.e. $\times 10^{-6}$)
n	nano	divide by 1 000 000 000	(i.e. $\times 10^{-9}$)
p	pico	divide by 1 000 000 000 000	(i.e. $\times 10^{-12}$)

1.4 Efficiency

It should be noted that when a device converts or transforms energy, some of the input energy is consumed to make the device operate. The **efficiency of this operation is defined** as

$$\text{Efficiency} = \frac{\text{energy output in a given time}}{\text{energy input in the same time}} = \frac{W_o}{W_{in}}$$

$$= \frac{\text{power output}}{\text{power input}} = \frac{P_o}{P_{in}}$$

Efficiency Symbol: η Unit: **none**

$$\therefore \quad \eta = \frac{P_o}{P_{in}} \qquad [1.9]$$

Example 1.6 **A generating station has a daily output of 280 MW h and uses 500 t (tonnes) of coal in the process. The coal releases 7 MJ/kg when burnt. Calculate the overall efficiency of the station.**

Input energy per day is

$$W_{in} = 7 \times 10^6 \times 500 \times 1000$$
$$= 35.0 \times 10^{11} \text{ J}$$

Output energy per day is

$$W_o = 280 \text{ MW h}$$
$$= 280 \times 10^6 \times 3.6 \times 10^3 = 10.1 \times 10^{11} \text{ J}$$

$$\eta = \frac{W_o}{W_{in}} = \frac{10.1 \times 10^{11}}{35.0 \times 10^{11}} = \mathbf{0.288}$$

Example 1.7 **A lift of 250 kg mass is raised with a velocity of 5 m/s. If the driving motor has an efficiency of 85 per cent, calculate the input power to the motor.**

Weight of lift is

$$F = mg = 250 \times 9.81 = 2452 \text{ N}$$

Output power of motor is

$$P_o = Fu = 2452 \times 5 = 12\,260 \text{ W}$$

Input power to motor is

$$P_{in} = \frac{P_o}{\eta} = \frac{12\,260}{0.85} = 14\,450 \text{ W} = \mathbf{14.5 \text{ kW}}$$

1.5 Sources of Electrical Power

1) Coal
2) Natural gas
3) Hydro
4) Nuclear
5) Wind
6) Oil
7) Solar
8) Biofuels
9) Batteries: Primary cells (thrown away after use) and Secondary cells (rechargeable e.g. car batteries)

1.6 Summary of important formulae

F [newtons] $= m$ [kilograms] $\times a$ [metres per second squared] [1.1]

i.e. $F = ma$

Torque $T = Fr$ (newton metres) [1.3]

Work $W = Fl$ (joules) [1.4]

Work = Energy

Kinetic energy $W = \frac{1}{2}mu^2$ [1.5]

Power $P = Fu$ (watts) [1.7]

$= T\omega = M\omega = 2\pi nT$ [1.8]

Efficiency $\eta = P_o / P_{in}$ [1.9]

1.7 Summary of terms, units and their symbols

Quantity	Quantity Symbol	Unit	Unit symbol
Length	l	metre	m
Mass	m	kilogram	kg
Time	t	second	s
Velocity	v	metres per second	m/s or m s^{-1}
Acceleration	a	metres per second squared	m/s^2 or m s^{-2}
Force	F	newton	N
Electrical charge or quantity	Q	coulomb	C
Electric current	I	ampere	A
Resistance	R	ohm	Ω
Conductance	G	siemen	S
Electromotive force	E	volt	V
Potential difference	V	volt	V
Work	W	joule	J
Energy	E (or W)	joule	J
Power	P	watt	W

1.8 Greek Alphabet and Common Prefixes

Greek alphabet

LETTER	*UPPER CASE*	*LOWER CASE*
Alpha	A	α
Beta	B	β
Gamma	Γ	γ
Delta	Δ	δ
Epsilon	E	ε
Zeta	Z	ζ
Eta	H	η
Theta	Θ	θ
Iota	I	ι
Kappa	K	κ
Lambda	Λ	λ
Mu	M	μ
Nu	N	ν
Xi	Ξ	ξ
Omicron	O	o
Pi	Π	π
Rho	P	ρ
Sigma	Σ	σ
Tau	T	τ
Upsilon	Υ	υ
Phi	Φ	ϕ
Chi	X	χ
Psi	Ψ	ψ
Omega	Ω	ω

Common prefixes

PREFIX	*NAME*	*MEANING: multiply by*
E	exa	10^{18}
P	peta	10^{15}
T	tera	10^{12}
G	giga	10^{9}
M	mega	10^{6}
k	kilo	10^{3}
h	hecto	10^{2}
da	deca	10^{1}
d	deci	10^{-1}
c	centi	10^{-2}
m	milli	10^{-3}
μ	micro	10^{-6}
n	nano	10^{-9}
p	pico	10^{-12}
f	femto	10^{-15}
a	atto	10^{-18}

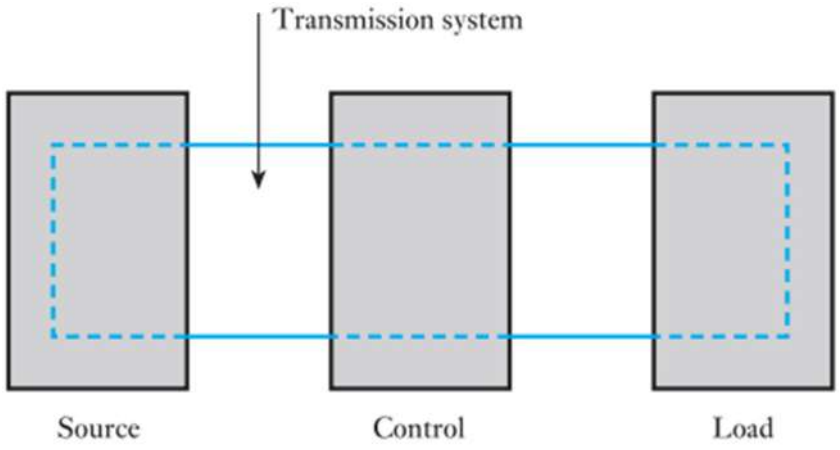
2 Introduction to Electrical Systems

Electricity can be considered from two points of view. The **scientist** is concerned with what happens in an electric system and seeks to explain its mysteries. The **engineer** accepts that electricity is there and seeks to make use of its properties without the need to fully understand them.

In fact, electrical engineering could be summarized into **four categories**:

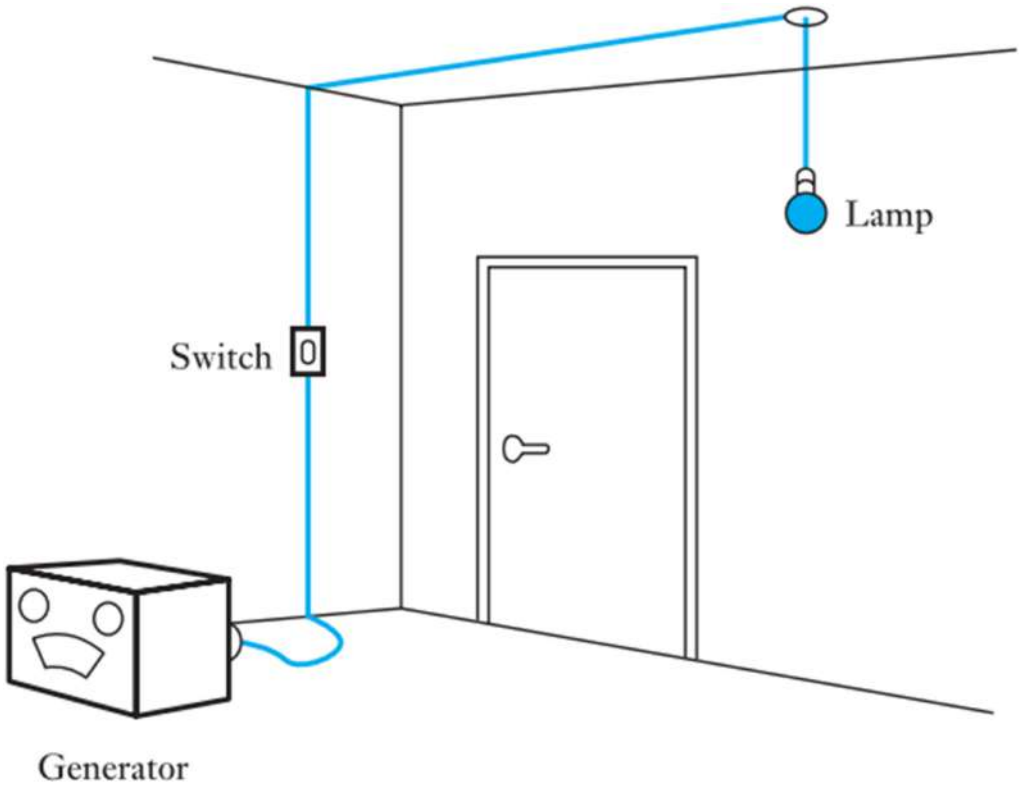
1) The production of electrical energy.
2) The transmission of electrical energy.
3) The application of electrical energy.
4) The control of electrical energy.

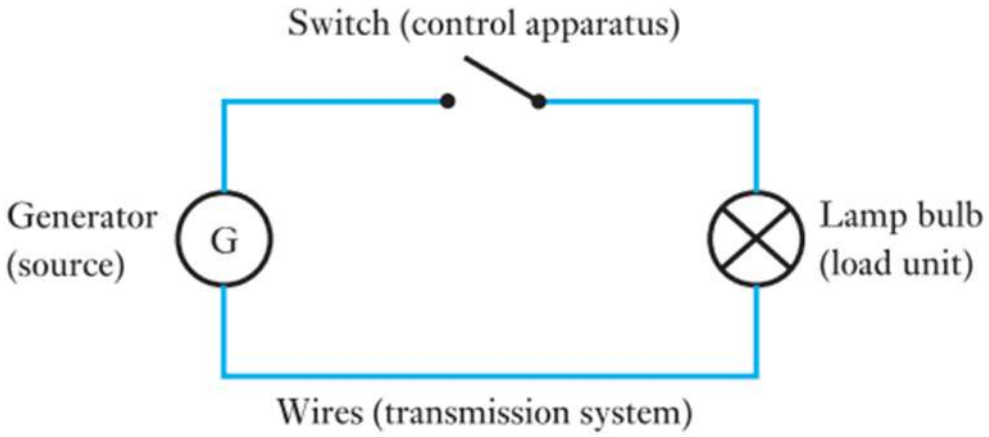
A **basic electrical system has four constituent** parts as shown below:



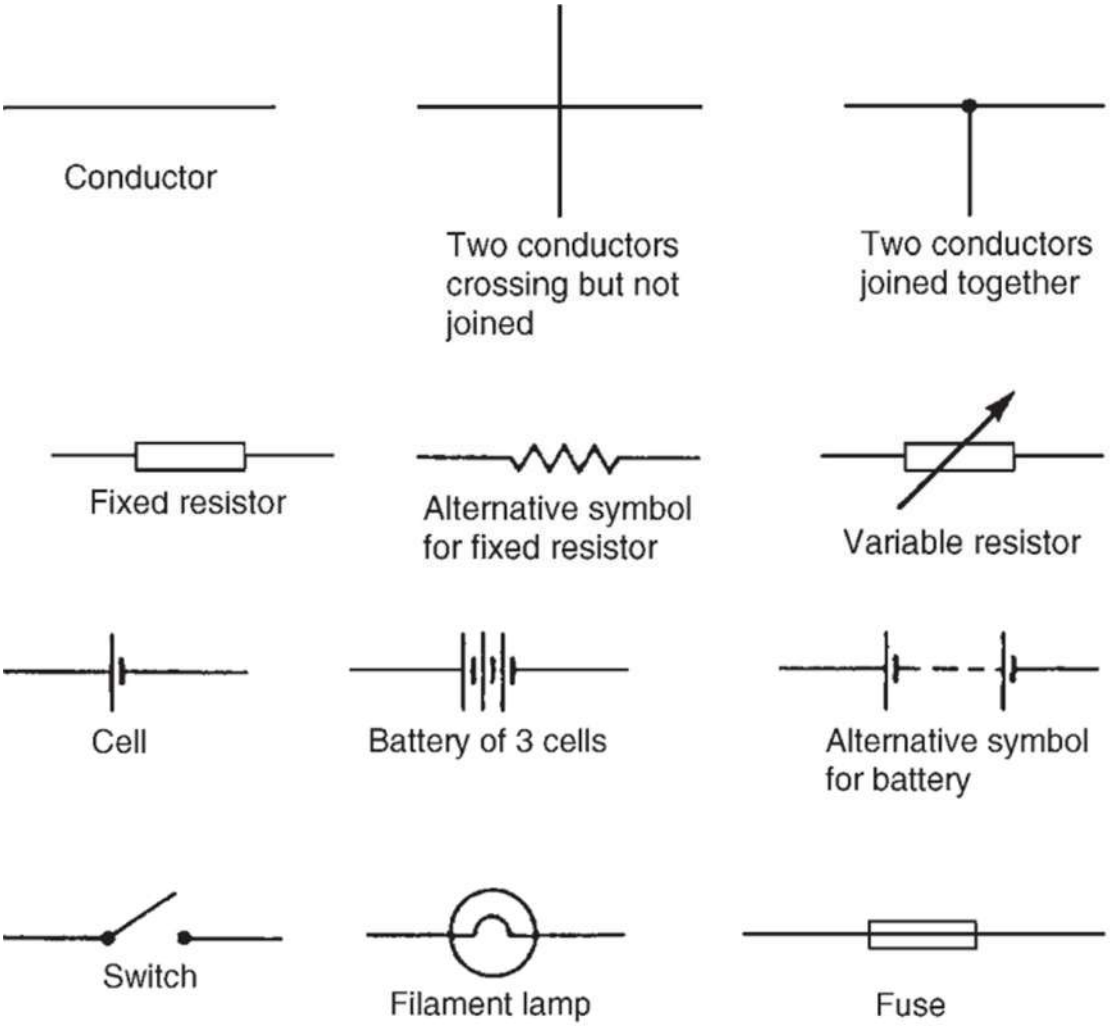
1. *The source*. The function of the source is to provide the energy for the electrical system. A source may usually be thought of as a battery or a generator, although for simplicity we might even think of a socket outlet as a source.
2. *The load*. The function of the load is to absorb the electrical energy supplied by the source. Most domestic electrical equipment constitutes loads. Common examples include lamps and heaters, all of which accept energy from the system.
3. *The transmission system*. This conducts the energy from the source to the load. Typically the transmission system consists of insulated wire.
4. *The control apparatus*. As the name suggests, its function is to control. The most simple control is a switch which permits the energy to flow or else interrupts the flow.

Consider a simple situation with which you are familiar – the **electric light in our room**.





Electrical systems involve the use of electric circuits. **Symbols are used for components in electrical circuit diagrams and some of the more common ones are shown below:**



Symbol	Representing
	Wires (conductors)
G	Generator
	Lamp bulb
	Switch

2.1 Electric current and quantity of electricity

All **atoms** consist of **protons, neutrons** and **electrons**. The **protons, which have positive electrical charges, and the neutrons, which have no electrical charge**, are contained within the **nucleus**. Removed from the nucleus are minute **negatively charged particles called electrons.** Atoms of different materials differ from one another by having different numbers of protons, neutrons and electrons. An equal number of protons and electrons exist within an atom and it is said to be electrically balanced, as the positive and negative charges cancel each other out. When there are more than two electrons in an atom the electrons are arranged into shells at various distances from the nucleus. Electrons that move from one atom to another are called **free electrons** and such random motion can continue indefinitely.

However, if an electric pressure or voltage is applied across any material there is a tendency for electrons to move in a particular direction.

When an atom loses an electron, it becomes a positively charged ion (cation) while it becomes a negatively charged ion (anion) if it gains an electron. Protons and cations make up positive charges while electrons and anions make up negative charges. Electricity is the rate of flow of charge.

Electricity appears in one of two forms which, by convention, are called negative and positive electricity. Electric charge is the excess of negative or positive electricity on a body or in space. If the excess is negative, the body is said to have a negative charge and vice versa.

A body containing a number of ionized atoms is also said to be electrically charged. It can be shown that positively and negatively charged bodies are mutually attracted to one another whereas similarly charged bodies repel one another.

2.1.1 Movement of electrons

All electrons have a certain potential energy. Given a suitable medium in which to exist, they move freely from one energy level to another and this movement, when undertaken in a concerted manner, is termed an electric current flow. Conventionally it is said that the current flows from a point of high energy level to a point of low energy level. These points are said to have high potential and low potential respectively. For convenience the point of high potential is termed the positive and the point of low potential is termed the negative, hence conventionally a current is said to flow from positive to negative. This convention was in general use long before the nature of electric charge was discovered. Unfortunately it was found that electrons move in the other direction since the negatively charged electron is attracted to the positive potential. Thus conventional current flows in the opposite direction to that of electron current. Normally only conventional current is described by the term current.

The transfer of electrons takes place more readily in a medium in which atoms can readily release electrons, e.g. copper, aluminium, silver. Such a material is termed a **conductor**. A material that does not readily permit electron flow is termed an **insulator**, e.g. porcelain,

nylon, rubber. There is also a family of materials termed **semiconductors** which have certain characteristics that belong to neither of the other groups.

Conductors	Insulators
Copper	Glass
Aluminium	Rubber
Silver	Plastic
Platinum	Air
Bronze	Varnish
Gold	Paper
	Wood
	Mica
	Ceramic
	Certain oils

The unit used to measure the **quantity of electrical charge, Q,** is called the **coulomb C** (where 1 coulomb = $6.24*10^{18}$ electrons). Generally, if ***I*** is the current in amperes and ***t*** the time in seconds during which the current flows:

$$Q = I \times t \text{ coulombs}$$

Problem 1. What current must flow if 0.24 coulombs is to be transferred in 15 ms?

Since the quantity of electricity, $Q = It$, then

$$I = \frac{Q}{t} = \frac{0.24}{15 \times 10^{-3}} = \frac{0.24 \times 10^{3}}{15} = \frac{240}{15} = \mathbf{16\ A}$$

Problem 2. If a current of 10 A flows for four minutes, find the quantity of electricity transferred.

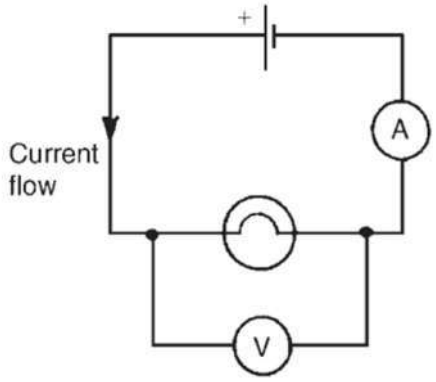
Quantity of electricity, $Q = It$ coulombs

$I = 10$ A; $t = 4 \times 60 = 240$ s

Hence $Q = 10 \times 240 = \mathbf{2400\ C}$

For a continuous current to flow between two points in a circuit, a **potential difference (p.d.)** or **voltage**, ***V***, is required between them; a complete conducting path is necessary to and from the source of electrical energy. The unit of p.d. is the **volt**, **V.**

Consider the figure below where a cell is connected across a filament lamp. Current flow, by convention, is considered as flowing from the positive terminal of the cell, around the circuit to the negative terminal.



The flow of electric current is subject to friction. This friction, or opposition, is called **resistance *R*** and is the property of a conductor that limits current. The unit of resistance is the **ohm (Ω).**

$$\textbf{resistance } R = \frac{\textbf{potential difference}}{\textbf{current}}$$

2.1.2 Ohm's law.

Ohm's law states that the current I flowing in a circuit is directly proportional to the applied voltage V and inversely proportional to the resistance R, provided the temperature remains constant.

$$I = \frac{V}{R} \text{ or } V = IR \text{ or } R = \frac{V}{I}$$

2.1.3 Resistors

A resistor is a device which provides resistance in an electrical circuit. The resistance of a resistor is said to be linear if the current through the resistor is proportional to the p.d. across its terminals. If the resistance were to vary with the magnitude of either the voltage or the current, the resistor is said to be non-linear. Resistors made from semiconductor materials are examples of non-linear resistors.

In this course resistors may be assumed linear, i.e. their resistance will be assumed to remain constant when the temperature is maintained constant. We cannot ignore the effect of temperature since all resistors dissipate heat when operating, i.e. if a resistor passes a current I, then energy is brought into the resistor at the rate of I^2R. In order to release this heat energy, the resistor must become warmer than its surroundings until it can release the heat energy at the same rate as it is arriving. Therefore we have to assume that the effect of this temperature rise is negligible. All resistors have a power rating which is the maximum power that can be dissipated without the temperature rise being such that damage occurs to the resistor. Thus a 1 W resistor with a resistance of 100 Ω can pass a current of 100 mA.

Conductor wires and cables are similarly rated. Although we like to assume that a conductor has no resistance, in fact all have some resistance. The passing of a current therefore causes the conductor to heat, and if the heating effect is too great the insulant material can be damaged. The rating is therefore determined by the temperature which the insulant can withstand.

Therefore if we wish to purchase a resistor, we require to specify not only the required resistance but also the power rating. In electronic circuits, the common standard rating are 0.25W, 0.5W, 1 W and 2 W. Thus if we required a resistor to dissipate 1.1 W, we would select a 2 W resistor since the rating must exceed the operational value. In power circuits, much higher ratings up to a megawatt and more can be experienced, but such operational conditions are expensive since we would be paying for energy which we are throwing away. Thus power engineers avoid the use of resistors as much as possible, seeking to utilize energy with minimal loss. In many cases, it is normally assumed that the power ratings have been correctly specified, and therefore resistors will only have their resistances given.

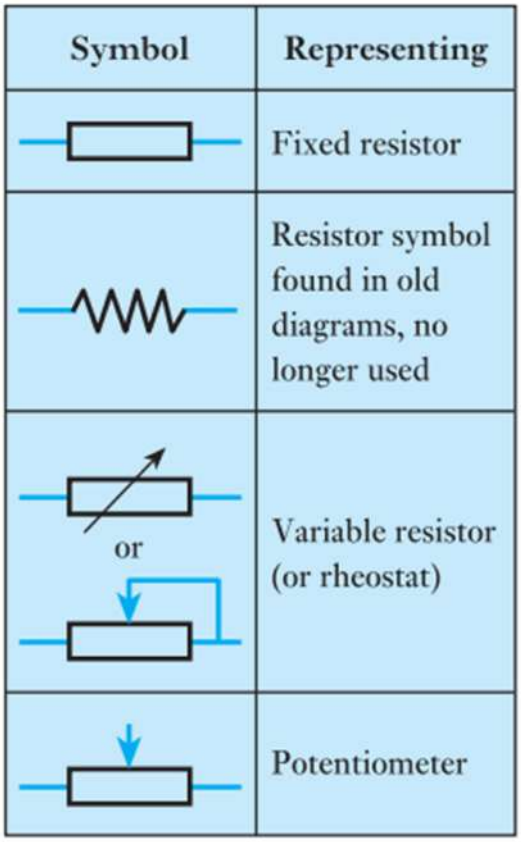


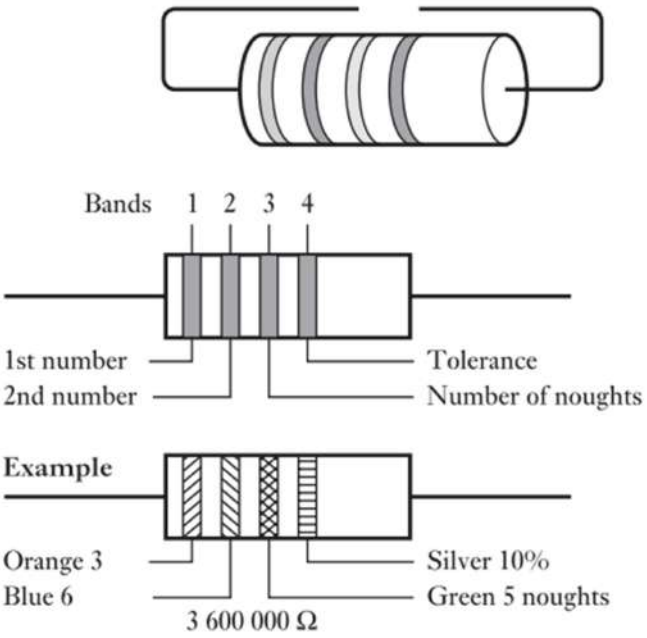
Fig. 2.9 Resistor types and symbols

Potentiometer – a three terminal device used for voltage control.

Rheostat – a two terminal device used current control.

2.1.4 Resistor coding

It is usual to identify the resistor ratings by means of rings painted around the resistors. One of the bands is always placed near to the end of the resistor and should be taken as the first band. The first, second and third bands are used to indicate the resistance of the resistor by means of a colour code:



Digit	Colour
0	Black
1	Brown
2	Red
3	Orange
4	Yellow
5	Green
6	Blue
7	Violet
8	Grey
9	White
Tolerance	**Colour**
5%	Gold
10%	Silver
20%	No colour band

The application of this code is best explained by the example shown in the figure above. Here the first two bands are orange and blue which, from the table, are 3 and 6 respectively. Therefore we are being told that the resistance has a numerical value of 36. The third band tells us how many zeros to put after that number. In this case, the third band is green and there should be five zeros, i.e. the resistance is 3 600 000 Ω.

The fourth band is the tolerance which is the extent to which the actual value of the resistance can vary relative to nominal value indicated by the first three bands. Thus for the resistor shown in the Figure above, the fourth band is silver, which indicates that the resistance has a value between 3 600 000 + 10 per cent and 3 600 000 - 10 per cent. Ten per cent of 3 600 000 is 360 000. Therefore the resistance will be somewhere between 3 960 000 and 3 240 000 Ω.

Some resistors have a fifth band which indicates a reliability factor which is the percentage of failure per 1000 h of use. For instance, a 1 per cent failure rate would suggest that one from every hundred resistors will not remain with tolerance after 1000 h of use. Band 5 colours indicate the following percentages:

1	Brown
0.1	Red
0.01	Orange
0.001	Yellow

A list of readily available standard values appears in the Table below. All are available with 5 per cent tolerance whereas those in bold type are also available with 10 per cent tolerance.

Ohms (Ω)					Kilohms (kΩ)		Megohms (MΩ)	
0.10	**1.0**	**10**	**100**	**1000**	**10**	**100**	**1.0**	**10.0**
0.11	1.1	11	110	1100	11	110	1.1	11.0
0.12	**1.2**	**12**	**120**	**1200**	**12**	**120**	**1.2**	**12.0**
0.13	1.3	13	130	1300	13	130	1.3	13.0
0.15	**1.5**	**15**	**150**	**1500**	**15**	**150**	**1.5**	**15.0**
0.16	1.6	16	160	1600	16	160	1.6	16.0
0.18	**1.8**	**18**	**180**	**1800**	**18**	**180**	**1.8**	**18.0**
0.20	2.0	20	200	2000	20	200	2.0	20.0
0.22	**2.2**	**22**	**220**	**2200**	**22**	**220**	**2.2**	**22.0**
0.24	2.4	24	240	2400	24	240	2.4	
0.27	**2.7**	**27**	**270**	**2700**	**27**	**270**	**2.7**	
0.30	3.0	30	300	3000	30	300	3.0	
0.33	**3.3**	**33**	**330**	**3300**	**33**	**330**	**3.3**	
0.36	3.6	36	360	3600	36	360	3.6	
0.39	**3.9**	**39**	**390**	**3900**	**39**	**390**	**3.9**	
0.43	4.3	43	430	4300	43	430	4.3	
0.47	**4.7**	**47**	**470**	**4700**	**47**	**470**	**4.7**	
0.51	5.1	51	510	5100	51	510	5.1	
0.56	**5.6**	**56**	**560**	**5600**	**56**	**560**	**5.6**	
0.62	6.2	62	620	6200	62	620	6.2	
0.68	**6.8**	**68**	**680**	**6800**	**68**	**680**	**6.8**	
0.75	7.5	75	750	7500	75	750	7.5	
0.82	**8.2**	**82**	**820**	**8200**	**82**	**820**	**8.2**	
0.91	9.1	91	910	9100	91	910	9.1	

Many resistors can be so small or of the wrong shape so that the colour coding is difficult to apply. Instead a letter code is used and this is best explained by examination of the following list:

Resistance	Marking
0.47 Ω	R47
4.7 Ω	4R7
47 Ω	47R
470 Ω	470R
4.7 kΩ	4K7
47 kΩ	47K
4.7 MΩ	4M7

2.1.5 Resistance and resistivity

The resistance of an electrical conductor depends on 4 factors, these being:

a). the length of the conductor,

b). the cross-sectional area of the conductor,

c). the type of material and

d). the temperature of the material.

Resistance, R, is directly proportional to length, l, of a conductor, i.e. $R \propto l$. Thus, for example, if the length of a piece of wire is doubled, then the resistance is doubled.

Resistance, R, is inversely proportional to cross-sectional area, a, of a conductor, i.e. $R \propto 1/a$. Thus, for example, if the cross-sectional area of a piece of wire is doubled then the resistance is halved.

Since $R \propto l$ and $R \propto 1/a$ then $R \propto l/a$. By inserting a constant of proportionality into this relationship the type of material used may be taken into account. The constant of proportionality is known as the **resistivity** of the material and is given the symbol ρ (Greek rho). Thus,

resistance $\boxed{R = \dfrac{\rho l}{a} \text{ ohms}}$

ρ is measured in ohm metres (Ωm)

Resistivity varies with temperature and some typical values of resistivities measured at about room temperature are given below:

Copper 1.7×10^{-8} Ωm (or 0.017 $\mu\Omega$m)

Aluminium 2.6×10^{-8} Ωm (or 0.026 $\mu\Omega$m)

Carbon (graphite) 10×10^{-8} Ωm (or 0.10 $\mu\Omega$m)

Glass 1×10^{10} Ωm (or 10^4 $\mu\Omega$m)

Mica 1×10^{13} Ωm (or 10^7 $\mu\Omega$m)

Note that good conductors of electricity have a low value of resistivity and good insulators have a high value of resistivity.

Problem 1. The resistance of a 5 m length of wire is 600 Ω. Determine (a) the resistance of an 8 m length of the same wire, and (b) the length of the same wire when the resistance is 420 Ω.

(a) Resistance, R, is directly proportional to length, l, i.e. $R \propto l$

Hence, 600 $\Omega \propto 5$ m or $600 = (k)(5)$, where k is the coefficient of proportionality. Hence,

$$k = \frac{600}{5} = 120$$

When the length l is 8 m, then resistance

$$R = kl = (120)(8) = \mathbf{960\ \Omega}$$

(b) When the resistance is 420 Ω, $420 = kl$, from which,

$$\text{length } l = \frac{420}{k} = \frac{420}{120} = \mathbf{3.5\ m}$$

Problem 2. A piece of wire of cross-sectional area 2 mm^2 has a resistance of 300 Ω. Find (a) the resistance of a wire of the same length and material if the cross-sectional area is 5 mm^2, (b) the cross-sectional area of a wire of the same length and material of resistance 750 Ω

Resistance R is inversely proportional to cross-sectional area, a, i.e. $R \propto \frac{1}{a}$

Hence $300\ \Omega \propto \frac{1}{2\ mm^2}$ or $300 = (k)\left(\frac{1}{2}\right)$,

from which, the coefficient of proportionality, $k = 300 \times 2 = 600$

(a) When the cross-sectional area $a = 5\ mm^2$ then $R = (k)\left(\frac{1}{5}\right)$

$= (600)\left(\frac{1}{5}\right) = \mathbf{120\ \Omega}$

(Note that resistance has decreased as the cross-sectional is increased.)

(b) When the resistance is 750 Ω then $750 = (k)\,(1/a)$, from which

cross-sectional area, $a = \frac{k}{750} = \frac{600}{750} = \mathbf{0.8\ mm^2}$

Problem 3. A wire of length 8 m and cross-sectional area 3 mm^2 has a resistance of 0.16 Ω. If the wire is drawn out until its cross-sectional area is 1 mm^2, determine the resistance of the wire.

Resistance R is directly proportional to length l, and inversely proportional to the cross-sectional area, a, i.e.,

i.e., $R \propto \frac{l}{a}$ or $R = k\left(\frac{l}{a}\right)$, where k is the coefficient of proportionality.

Since $R = 0.16$, $l = 8$ and $a = 3$, then $0.16 = (k)\left(\frac{8}{3}\right)$, from which

$k = 0.16 \times \frac{3}{8} = 0.06$

If the cross-sectional area is reduced to $\frac{1}{3}$ of its original area then the length must be tripled to 3×8, i.e., 24 m

New resistance $R = k\left(\frac{l}{a}\right) = 0.06\left(\frac{24}{1}\right) = \mathbf{1.44\ \Omega}$

Problem 4. Calculate the resistance of a 2 km length of aluminium overhead power cable if the cross-sectional area of the cable is 100 mm^2. Take the resistivity of aluminium to be $0.03 \times 10^{-6}\ \Omega m$

Length $l = 2\ km = 2000$ m; area, $a = 100\ mm^2 = 100 \times 10^{-6} m^2$; resistivity $\rho = 0.03 \times 10^{-6}\ \Omega m$

$$\text{Resistance} \quad R = \frac{\rho l}{a} = \frac{(0.03 \times 10^{-6}\ \Omega m)(2000\ m)}{(100 \times 10^{-6}\ m^2)} = \frac{0.03 \times 2000}{100}\ \Omega$$
$$= \mathbf{0.6\ \Omega}$$

Problem 5. Calculate the cross-sectional area, in mm^2, of a piece of copper wire, 40 m in length and having a resistance of 0.25 Ω. Take the resistivity of copper as $0.02 \times 10^{-6}\ \Omega m$

Resistance $R = \frac{\rho l}{a}$ hence cross-sectional area $a = \frac{\rho l}{R}$

$$= \frac{(0.02 \times 10^{-6}\ \Omega m)(40\ m)}{0.25\ \Omega} = 3.2 \times 10^{-6}\ m^2$$
$$= (3.2 \times 10^{-6}) \times 10^{6}\ mm^2 = \mathbf{3.2\ mm^2}$$

Problem 6. The resistance of 1.5 km of wire of cross-sectional area 0.17 mm^2 is 150 Ω. Determine the resistivity of the wire.

Resistance, $R = \frac{\rho l}{a}$

hence, resistivity $\rho = \frac{Ra}{l} = \frac{(150\ \Omega)(0.17 \times 10^{-6}\ m^2)}{(1500\ m)}$

$$= 0.017 \times 10^{-6}\ \Omega\text{m or } 0.017\ \mu\Omega\text{m}$$

> Problem 7. Determine the resistance of 1200 m of copper cable having a diameter of 12 mm if the resistivity of copper is $1.7 \times 10^{-8}\ \Omega$m

Cross-sectional area of cable, $a = \pi r^2 = \pi\left(\frac{12}{2}\right)^2$

$$= 36\pi\ \text{mm}^2 = 36\pi \times 10^{-6}\ \text{m}^2$$

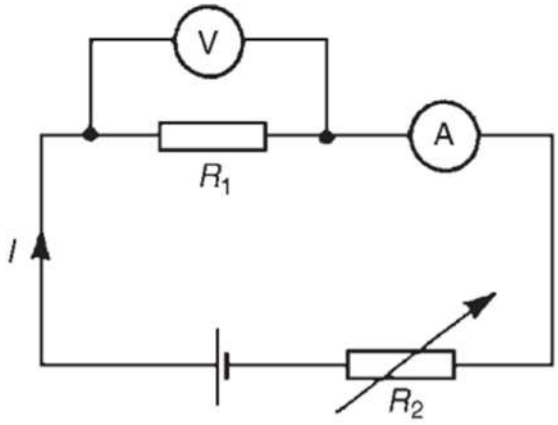
Resistance $R = \dfrac{\rho l}{a} = \dfrac{(1.7 \times 10^{-8}\ \Omega\text{m})\ (1200\ \text{m})}{(36\pi \times 10^{-6}\ \text{m}^2)}$

$$= \frac{1.7 \times 1200 \times 10^6}{10^8 \times 36\pi}\Omega = \frac{1.7 \times 12}{36\pi}\Omega$$

$$= \mathbf{0.180\ \Omega}$$

2.1.6 Basic electrical measuring instruments

An **ammeter** is an instrument used to measure current and must be connected **in series** with the circuit. Since all the current in the circuit passes through the ammeter it must have a very **low resistance**. A **voltmeter** is an instrument used to measure p.d. and must be connected **in parallel** with the part of the circuit whose p.d. is required. To avoid a significant current flowing through it a voltmeter must have a very **high resistance**. See the figure below:



An **ohmmeter** is an instrument for measuring resistance. A **multimeter**, or universal instrument, may be used to measure voltage, current and resistance.

The **cathode ray oscilloscope (CRO)** may be used to observe waveforms and to measure voltages and currents. The display of a CRO involves a spot of light moving across a screen.

2.1.7 Main effects of electric current

The three main effects of an electric current are:

(a) magnetic effect
(b) chemical effect
(c) heating effect

Some practical applications of the effects of an electric current include:

Magnetic effect:	bells, relays, motors, generators, transformers, telephones, car-ignition and lifting magnets
Chemical effect:	primary and secondary cells and electroplating
Heating effect:	cookers, water heaters, electric fires, irons, furnaces, kettles and soldering irons

2.1.8 Fuses

A **fuse** is used to prevent overloading of electrical circuits. The fuse, which is made of material having a low melting point, utilizes the heating effect of an electric current. A fuse is placed in an electrical circuit and if the current becomes too large the fuse wire melts and so breaks the circuit.

Problem 20. If 5 A, 10 A and 13 A fuses are available, state which is most appropriate for the following appliances which are both connected to a 240 V supply (a) Electric toaster having a power rating of 1 kW (b) Electric fire having a power rating of 3 kW

Power $P = VI$, from which, current $I = \dfrac{P}{V}$

(a) For the toaster, current $I = \dfrac{P}{V} = \dfrac{1000}{240} = \dfrac{100}{24} = 4\dfrac{1}{6}$ A

Hence a **5 A fuse** is most appropriate

(b) For the fire, current $I = \dfrac{P}{V} = \dfrac{3000}{240} = \dfrac{300}{24} = 12\dfrac{1}{2}$ A

Hence a **13 A fuse** is most appropriate

2.2 Revision Questions

$$Q = I \times t$$

1 In what time would a current of 10 A transfer a charge of 50 C? [5 s]

2 A current of 6 A flows for 10 minutes. What charge is transferred? [3600 C]

3 How long must a current of 100 mA flow so as to transfer a charge of 80 C? [13 min 20 s]

Ohm's law

4 The current flowing through a heating element is 5 A when a p.d. of 35 V is applied across it. Find the resistance of the element. [7 Ω]

5 A 60 W electric light bulb is connected to a 240 V supply. Determine (a) the current flowing in the bulb and (b) the resistance of the bulb. [(a) 0.25 A (b) 960 Ω]

6 Graphs of current against voltage for two resistors P and Q are shown in Figure 2.6. Determine the value of each resistor. [2 mΩ, 5 mΩ]

7 Determine the p.d. which must be applied to a 5 kΩ resistor such that a current of 6 mA may flow. [30 V]

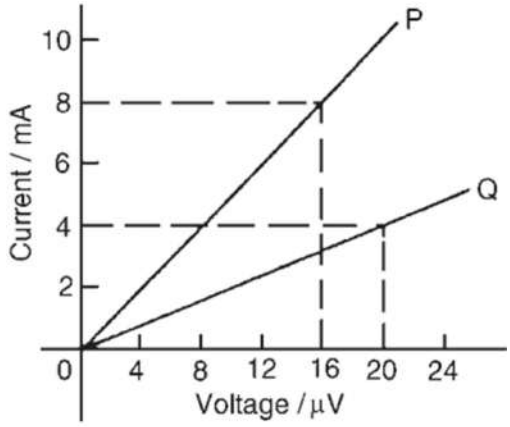


Figure 2.6

Power and energy

8 The hot resistance of a 250 V filament lamp is 625 Ω. Determine the current taken by the lamp and its power rating.
[0.4 A, 100 W]

9 Determine the resistance of a coil connected to a 150 V supply when a current of (a) 75 mA (b) 300 μA flows through it.
[(a) 2 kΩ (b) 0.5 MΩ]

10 Determine the resistance of an electric fire which takes a current of 12 A from a 240 V supply. Find also the power rating of the fire and the energy used in 20 h. [20 Ω, 2.88 kW, 57.6 kWh]

11 Determine the power dissipated when a current of 10 mA flows through an appliance having a resistance of 8 kΩ. [0.8 W]

12 85.5 J of energy are converted into heat in nine seconds. What power is dissipated? [9.5 W]

13 A current of 4 A flows through a conductor and 10 W is dissipated. What p.d. exists across the ends of the conductor? [2.5 V]

14 Find the power dissipated when:
   (a) a current of 5 mA flows through a resistance of 20 kΩ
   (b) a voltage of 400 V is applied across a 120 kΩ resistor
   (c) a voltage applied to a resistor is 10 kV and the current flow is 4 mA. [(a) 0.5 W (b) $1\frac{1}{3}$ W (c) 40 W]

15 A battery of e.m.f. 15 V supplies a current of 2 A for 5 min. How much energy is supplied in this time? [9 kJ]

16 In a household during a particular week three 2 kW fires are used on average 25 h each and eight 100 W light bulbs are used on average 35 h each. Determine the cost of electricity for the week if 1 unit of electricity costs 7p. [£12.46]

17 Calculate the power dissipated by the element of an electric fire of resistance 30 Ω when a current of 10 A flows in it. If the fire is on for 30 hours in a week determine the energy used. Determine also the weekly cost of energy if electricity costs 7.2p per unit.
[3 kW, 90 kWh, £6.48]

Fuses

18 A television set having a power rating of 120 W and electric lawn-mower of power rating 1 kW are both connected to a 240 V supply. If 3 A, 5 A and 10 A fuses are available state which is the most appropriate for each appliance. [3 A, 5 A]

Resistance and resistivity

1 The resistance of a 2 m length of cable is 2.5 Ω. Determine (a) the resistance of a 7 m length of the same cable and (b) the length of the same wire when the resistance is 6.25 Ω. [(a) 8.75 Ω (b) 5 m]

2 Some wire of cross-sectional area 1 mm^2 has a resistance of 20 Ω. Determine (a) the resistance of a wire of the same length and material if the cross-sectional area is 4 mm^2, and (b) the cross-sectional area of a wire of the same length and material if the resistance is 32 Ω. [(a) 5 Ω (b) 0.625 mm^2]

3 Some wire of length 5 m and cross-sectional area 2 mm^2 has a resistance of 0.08 Ω. If the wire is drawn out until its cross-sectional area is 1 mm^2, determine the resistance of the wire. [0.32 Ω]

4 Find the resistance of 800 m of copper cable of cross-sectional area 20 mm^2. Take the resistivity of copper as 0.02 μΩm. [0.8 Ω]

5 Calculate the cross-sectional area, in mm^2, of a piece of aluminium wire 100 m long and having a resistance of 2 Ω. Take the resistivity of aluminium as 0.03×10^{-6} Ωm. [1.5 mm^2]

6 (a) What does the resistivity of a material mean?

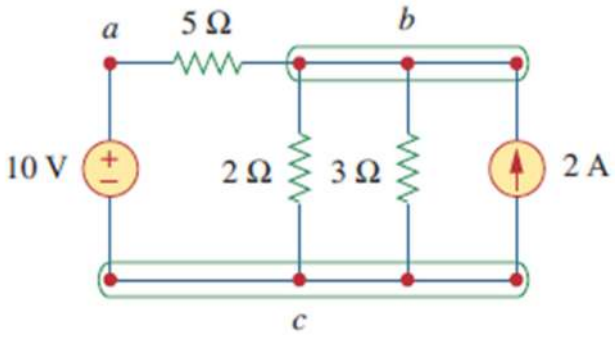
(b) The resistance of 500 m of wire of cross-sectional area 2.6 mm^2 is 5 Ω. Determine the resistivity of the wire in μΩm. [0.026 μΩm]

7 Find the resistance of 1 km of copper cable having a diameter of 10 mm if the resistivity of copper is 0.017×10^{-6} Ωm. [0.216 Ω]

3 Series and parallel networks

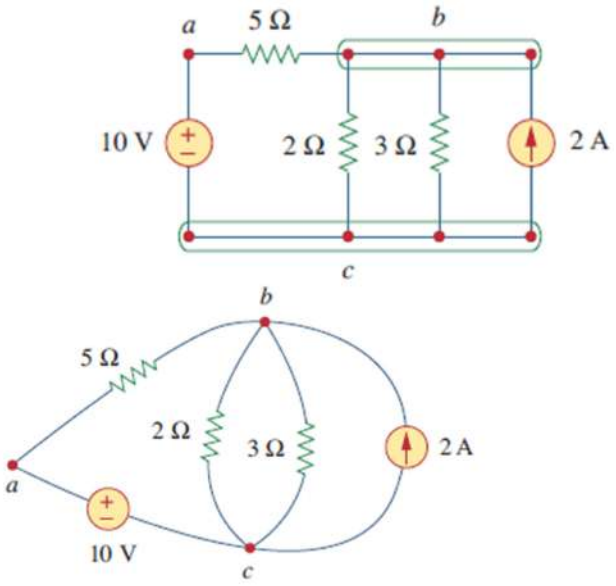
3.1 Nodes, Branches and Loops

Consider the figure shown below:



A **branch** represents a single element such as a voltage source or a resistor, i.e. a branch represents any two-terminal element. The circuit above has five branches, namely, the 10-V voltage source, the 2-A current source, and the three resistors.

A **node** is the point of connection between two or more branches. A node is usually indicated by **a dot** in a circuit. If a short circuit (a connecting wire) **connects two nodes, the two nodes constitute a single node.** The circuit given above has three nodes, a, b, and c. Notice that the three points that form node b are connected by perfectly conducting wires and therefore constitute a single point. The same is true of the four points forming node c. The figure above is redrawn below to illustrate the concept of nodes. The two circuits are identical.

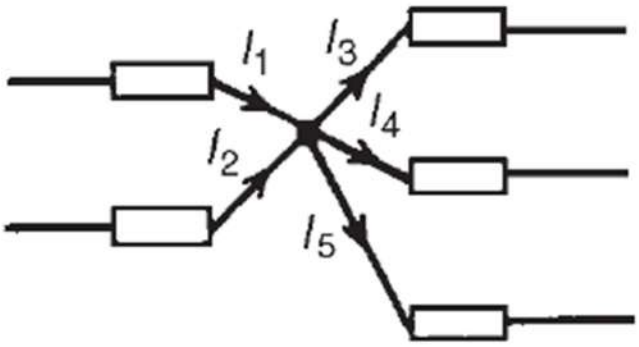


A **loop** is any closed path in a circuit. A loop is a **closed path** formed by starting at a node, passing through a set of nodes, and returning to the starting node without passing through any node **more than once.** A loop that does not contain any other loop within it is called **a mesh.**

3.2 Kirchhoff's laws

3.2.1 Kirchhoff's Current Law (KCL).

At any junction in an electric circuit the total current flowing towards that junction is equal to the total current flowing away from the junction, i.e. $\sum I = 0$



Therfore,

$$I_1 + I_2 = I_3 + I_4 + I_5 \quad \text{or} \quad I_1 + I_2 - I_3 - I_4 - I_5 = 0$$

This is also called ***Kirchhoff's first law which*** *states that the algebraic sum of all currents at a node is equal to zero.*

3.2.2 Kirchhoff's voltage law (KVL)

This is closely related to series circuits.

Kirchhoff's voltage law (KVL) states that the algebraic sum of all voltages around a closed path (or loop) is equal to zero.

Or

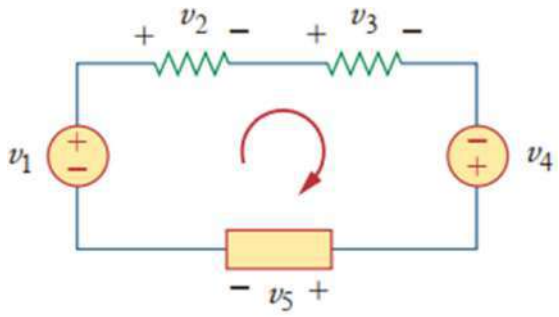
In any closed loop in a network, the algebraic sum of the voltage drops (i.e. products of current and resistance) taken around the loop is equal to the resultant e.m.f. acting in that loop.

This is also called **Kirchhoff's second law**. Expressed mathematically, KVL states that

$$\sum_{m=1}^{M} v_m = 0$$

where M is the number of voltages in the loop (or the number of branches in the loop) and v_m is the mth voltage.

To illustrate KVL, consider the circuit below.



The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop. We can start with any branch and go around the loop either clockwise or counterclockwise. Suppose we start with the voltage source and go clockwise around the loop as shown; then voltages would be $-v_1$, $+v_2$, $+v_3$, $-v_4$, and $+v_5$ in that order. For example, as we reach branch 3, the positive terminal is met first; hence, we have $+v_3$. For branch 4, we reach the negative terminal first; hence we $-v_4$. Thus, KVL yields

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

Rearranging terms gives

$$v_2 + v_3 + v_5 = v_1 + v_4$$

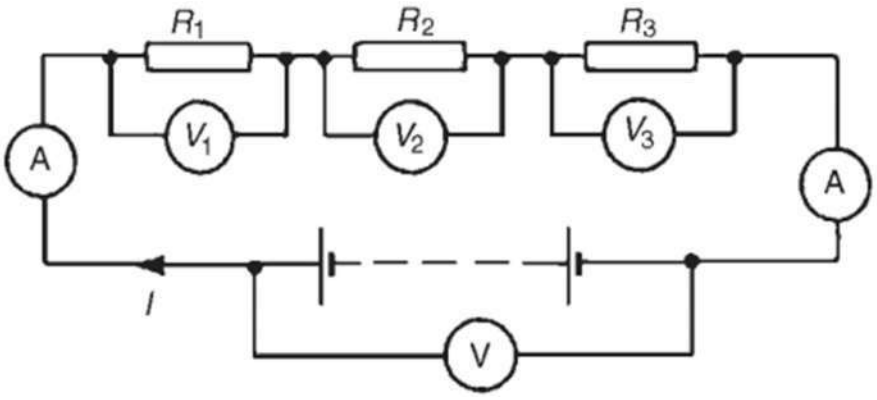
which may be interpreted as

Sum of voltage drops = Sum of voltage rises

NB: KVL can be applied in two ways: by taking either a clockwise or a counterclockwise trip around the loop. Either way, the algebraic sum of voltages around the loop is zero. Notice that if we had travelled counterclockwise, the result would have been $+v_1$, $-v_5$, $+v_4$, $-v_3$, and $-v_2$ which is the same as before except that the signs are reversed.

3.3 Series circuits

Consider the figure below:



In a series circuit:

(a) the current I is the same in all parts of the circuit and hence the same reading is found on each of the two ammeters shown, and
(b) the sum of the voltages V_1, V_2 and V_3 is equal to the total applied voltage, V, i.e.

$$\boxed{V = V_1 + V_2 + V_3}$$

That is, when voltage sources are connected in series, KVL can be applied to obtain the total voltage.

From Ohm's law:

$$V_1 = IR_1,\ V_2 = IR_2,\ V_3 = IR_3 \text{ and } V = IR$$

where R is the total circuit resistance.

Since $V = V_1 + V_2 + V_3$

then $IR = IR_1 + IR_2 + IR_3$

Dividing throughout by I gives

$$\boxed{R = R_1 + R_2 + R_3}$$

Thus for a series circuit, the total resistance is obtained by adding together the values of the separate resistances.

Problem 1. For the circuit shown in Figure 5.2, determine (a) the battery voltage V, (b) the total resistance of the circuit, and (c) the values of resistance of resistors R_1, R_2 and R_3, given that the p.d.'s across R_1, R_2 and R_3 are 5 V, 2 V and 6 V respectively.

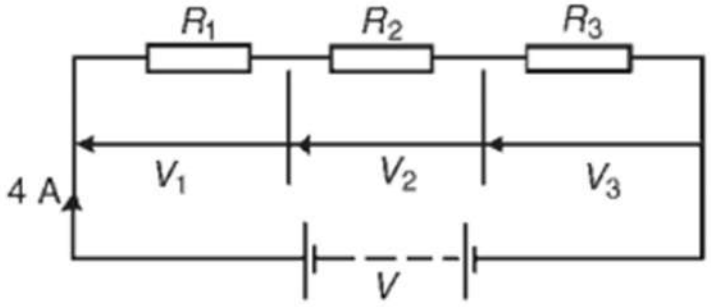


Figure 5.2

(a) Battery voltage $V = V_1 + V_2 + V_3$

$= 5 + 2 + 6 = \mathbf{13\ V}$

(b) Total circuit resistance $R = \dfrac{V}{I} = \dfrac{13}{4} = \mathbf{3.25\ \Omega}$

(c) Resistance $R_1 = \dfrac{V_1}{I} = \dfrac{5}{4} = \mathbf{1.25\ \Omega}$

Resistance $R_2 = \dfrac{V_2}{I} = \dfrac{2}{4} = \mathbf{0.5\ \Omega}$

Resistance $R_3 = \dfrac{V_3}{I} = \dfrac{6}{4} = \mathbf{1.5\ \Omega}$

(Check: $R_1 + R_2 + R_3 = 1.25 + 0.5 + 1.5 = 3.25\ \Omega = R$)

Problem 2. For the circuit shown in Figure 5.3, determine the p.d. across resistor R_3. If the total resistance of the circuit is 100 Ω, determine the current flowing through resistor R_1. Find also the value of resistor R_2

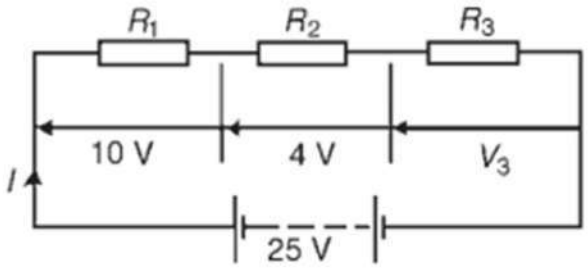


Figure 5.3

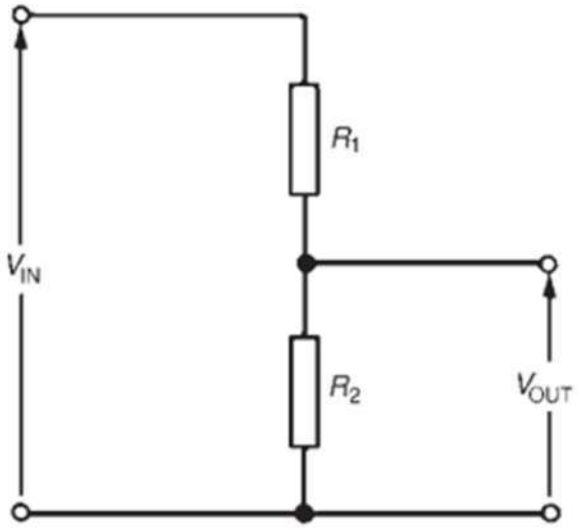
P.d. across R_3, $V_3 = 25 - 10 - 4 = \mathbf{11\ V}$

Current $I = \dfrac{V}{R} = \dfrac{25}{100} = \mathbf{0.25\ A}$, which is the current flowing in each resistor

Resistance $R_2 = \dfrac{V_2}{I} = \dfrac{4}{0.25} = \mathbf{16\ \Omega}$

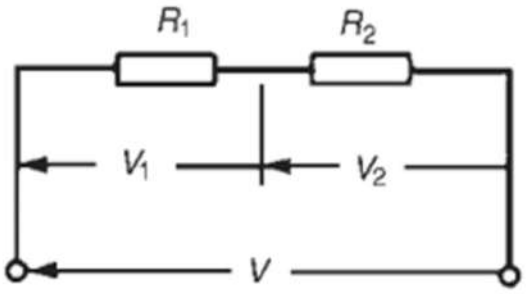
3.3.1 Potential divider / Potential division

The circuit shown below is often referred to as a **potential divider** circuit.



$$V_{OUT} = \left(\frac{R_2}{R_1 + R_2}\right) V_{IN}$$

Such a circuit can consist of a number of similar elements in series connected across a voltage source, voltages being taken from connections between the elements. Consequently, the voltage distribution for the circuit shown below is given by:

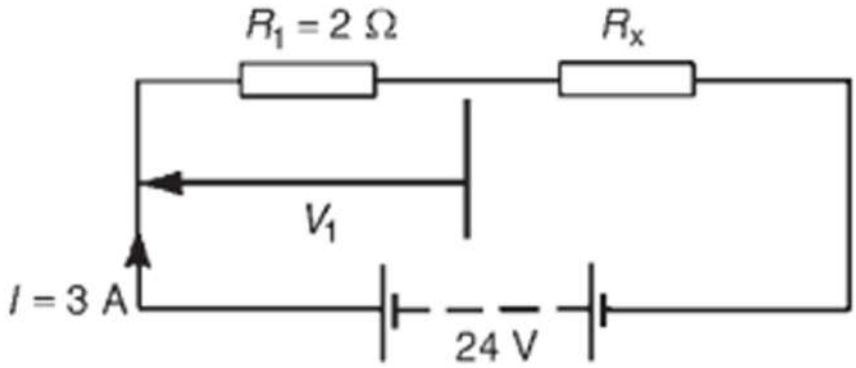


$$V_1 = \left(\frac{R_1}{R_1 + R_2}\right) \mathrm{V}$$

$$V_2 = \left(\frac{R_2}{R_1 + R_2}\right) \mathrm{V}$$

Problem 5. Two resistors are connected in series across a 24 V supply and a current of 3 A flows in the circuit. If one of the resistors has a resistance of 2 Ω determine (a) the value of the other resistor, and (b) the p.d. across the 2 Ω resistor. If the circuit is connected for 50 hours, how much energy is used?

The circuit diagram is shown in below:



(a) Total circuit resistance $R = \dfrac{V}{I} = \dfrac{24}{3} = 8\ \Omega$

Value of unknown resistance, $R_x = 8 - 2 = \mathbf{6\ \Omega}$

(b) P.d. across 2 Ω resistor, $V_1 = IR_1 = 3 \times 2 = \mathbf{6\ V}$
Alternatively, from above,

$$V_1 = \left(\frac{R_1}{R_1 + R_x}\right)\ \text{V} = \left(\frac{2}{2+6}\right)(24) = 6\ \text{V}$$

Energy used = power × time

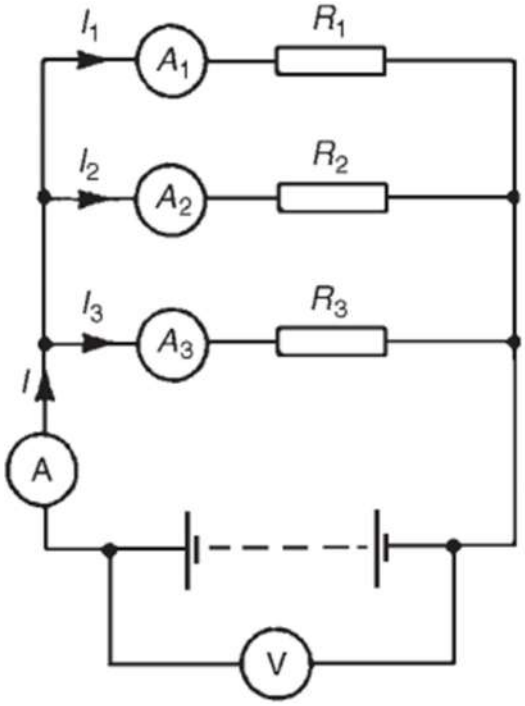
$= V \times I \times t$

$= (24 \times 3\ \text{W})(50\ \text{h})$

$= 3600\ \text{Wh} = \mathbf{3.6\ kWh}$

3.4 Parallel Circuits

The figure below shows three resistors, R_1, R_2 and R_3 connected across each other, i.e., in parallel, across a battery source of V volts.



When branches are connected in parallel, KCL can be applied to obtain the total current.

That is, in a parallel circuit:

(a) the sum of the currents I_1, I_2 and I_3 is equal to the total circuit current, I, i.e. $\boldsymbol{I = I_1 + I_2 + I_3}$, and
(b) the source p.d., V volts, is the same across each of the resistors.

From Ohm's law:

$$I_1 = \frac{V}{R_1},\ I_2 = \frac{V}{R_2},\ I_3 = \frac{V}{R_3} \text{ and } I = \frac{V}{R}$$

where R is the total circuit resistance.

Since $I = I_1 + I_2 + I_3$

then, $\dfrac{V}{R} = \dfrac{V}{R_1} + \dfrac{V}{R_2} + \dfrac{V}{R_3}$

Dividing throughout by V gives:

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

This equation must be used when finding the total resistance R of a parallel circuit. For the special case of **two resistors in parallel**

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1 R_2}$$

Hence $$\boxed{R = \frac{R_1 R_2}{R_1 + R_2}} \qquad \left(\text{i.e. } \frac{\text{product}}{\text{sum}}\right)$$

Problem 8. For the circuit shown in Figure 5.12, find (a) the value of the supply voltage V and (b) the value of current I.

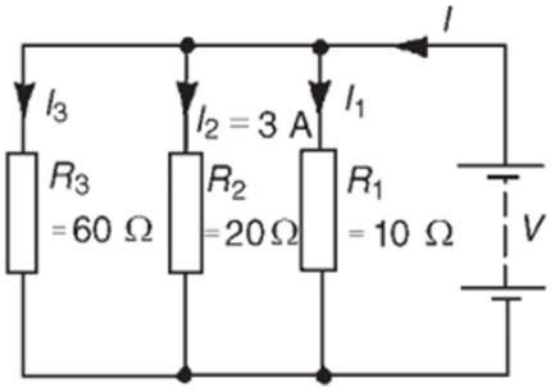


Figure 5.12

(a) P.d. across 20 Ω resistor $= I_2R_2 = 3 \times 20 = 60$ V, hence supply voltage $\boldsymbol{V = 60}$ **V** since the circuit is connected in parallel.

(b) Current $I_1 = \dfrac{V}{R_1} = \dfrac{60}{10} = 6$ A; $I_2 = 3$ A

$$I_3 = \frac{V}{R_3} = \frac{60}{60} = 1 \text{ A}$$

Current $I = I_1 + I_2 + I_3$ and hence $I = 6 + 3 + 1 = \mathbf{10\ A}$

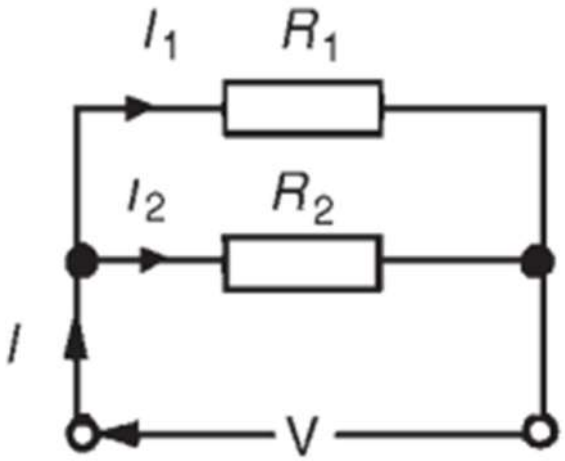
Alternatively, $\dfrac{1}{R} = \dfrac{1}{60} + \dfrac{1}{20} + \dfrac{1}{10} = \dfrac{1+3+6}{60} = \dfrac{10}{60}$

Hence total resistance $R = \dfrac{60}{10} = 6\ \Omega$

Current $I = \dfrac{V}{R} = \dfrac{60}{6} = \mathbf{10\ A}$

3.4.1 Current divider / Current division

Consider the figure below:



The total circuit resistance, R_T is given by:

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

and $V = IR_T = I\left(\frac{R_1 R_2}{R_1 + R_2}\right)$

Current $I_1 = \frac{V}{R_1} = \frac{I}{R_1}\left(\frac{R_1 R_2}{R_1 + R_2}\right) = \left(\frac{\boldsymbol{R_2}}{\boldsymbol{R_1 + R_2}}\right)\boldsymbol{(I)}$

Similarly,

current $I_2 = \frac{V}{R_2} = \frac{I}{R_2}\left(\frac{R_1 R_2}{R_1 + R_2}\right) = \left(\frac{\boldsymbol{R_1}}{\boldsymbol{R_1 + R_2}}\right)\boldsymbol{(I)}$

Summarizing, with reference to Figure 5.18

$$\boxed{\boldsymbol{I_1 = \left(\frac{R_2}{R_1 + R_2}\right)(I)}} \quad \textbf{and} \quad \boxed{\boldsymbol{I_2 = \left(\frac{R_1}{R_1 + R_2}\right)(I)}}$$

Problem 11. For the series-parallel arrangement shown in Figure 5.19, find (a) the supply current, (b) the current flowing through each resistor and (c) the p.d. across each resistor.

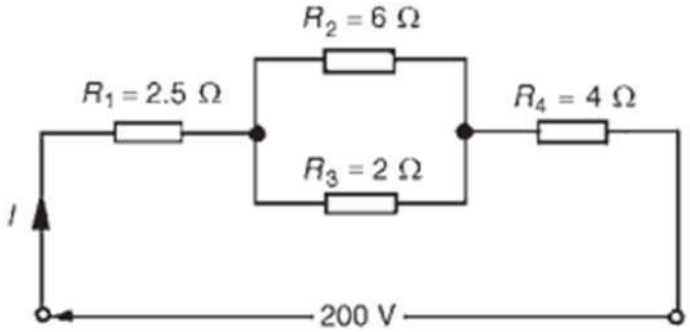


Figure 5.19

(a) The equivalent resistance R_x of R_2 and R_3 in parallel is:

$$R_x = \frac{6 \times 2}{6+2} = \frac{12}{8} = 1.5\ \Omega$$

The equivalent resistance R_T of R_1, R_x and R_4 in series is:

$$R_T = 2.5 + 1.5 + 4 = 8\ \Omega$$

$$\text{Supply current } I = \frac{V}{R_T} = \frac{200}{8} = \mathbf{25\ A}$$

(b) The current flowing through R_1 and R_4 is 25 A

$$\text{The current flowing through } R_2 = \left(\frac{R_3}{R_2 + R_3}\right) I = \left(\frac{2}{6+2}\right) 25$$

$$= \mathbf{6.25\ A}$$

$$\text{The current flowing through } R_3 = \left(\frac{R_2}{R_2 + R_3}\right) I = \left(\frac{6}{6+2}\right) 25$$

$$= \mathbf{18.75\ A}$$

(Note that the currents flowing through R_2 and R_3 must add up to the total current flowing into the parallel arrangement, i.e. 25 A)

(c) The equivalent circuit of Figure 5.19 is shown in Figure 5.20.

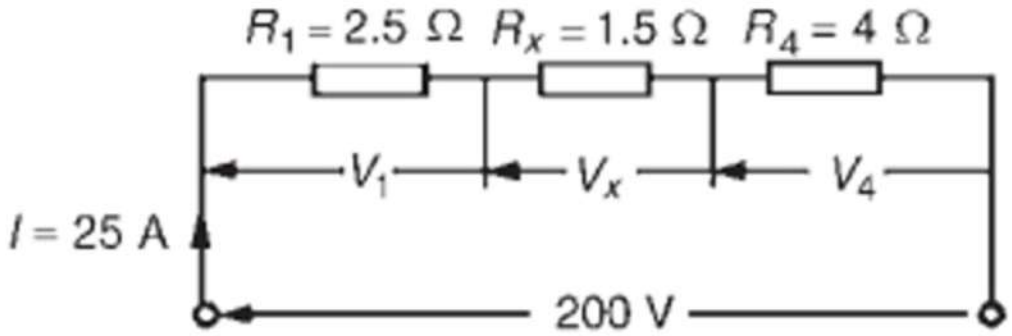


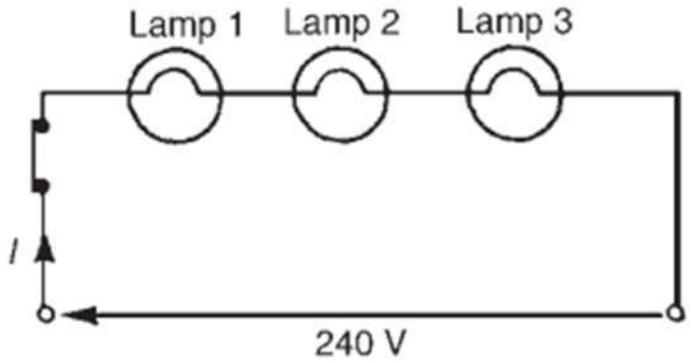
Figure 5.20

p.d. across R_1, i.e., $V_1 = IR_1 = (25)(2.5) = \mathbf{62.5\ V}$
p.d. across R_x, i.e., $V_x = IR_x = (25)(1.5) = \mathbf{37.5\ V}$
p.d. across R_4, i.e., $V_4 = IR_4 = (25)(4) = \mathbf{100\ V}$
Hence the p.d. across R_2 = p.d. across $R_3 = \mathbf{37.5\ V}$

3.5 Wiring lamps in series and in parallel

3.5.1.1 Series connection

The figure below shows three lamps, each rated at 240 V, connected in series across a 240 V supply.



(i) Each lamp has only $\frac{240}{3}$ V, i.e., 80 V across it and thus each lamp glows dimly.

(ii) If another lamp of similar rating is added in series with the other three lamps then each lamp now has $\frac{240}{4}$ V, i.e., 60 V across it and each now glows even more dimly.

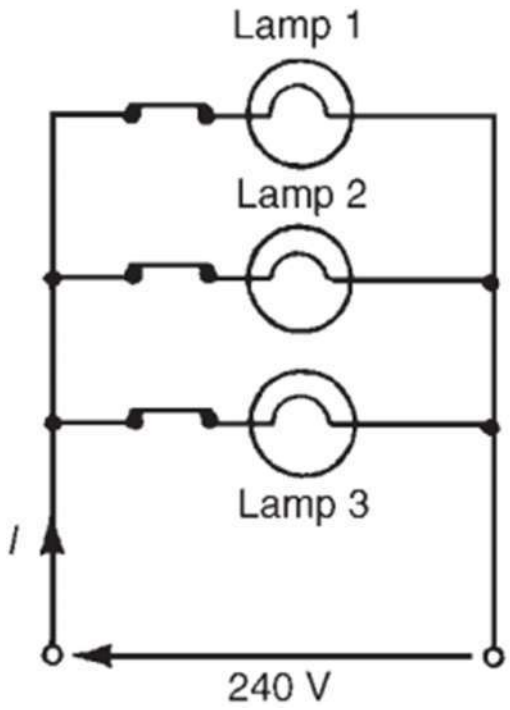
(iii) If a lamp is removed from the circuit or if a lamp develops a fault (i.e. an open circuit) or if the switch is opened then the circuit is broken, no current flows, and the remaining lamps will not light up.

(iv) Less cable is required for a series connection than for a parallel one.

The series connection of lamps is usually limited to decorative lighting such as for Christmas tree lights.

3.5.1.2 Parallel connection

The figure below shows three similar lamps, each rated at 240 V, connected in parallel across a 240 V supply.



(i) Each lamp has 240 V across it and thus each will glow brilliantly at their rated voltage.
(ii) If any lamp is removed from the circuit or develops a fault (open circuit) or a switch is opened, the remaining lamps are unaffected.
(iii) The addition of further similar lamps in parallel does not affect the brightness of the other lamps.
(iv) More cable is required for parallel connection than for a series one.

The parallel connection of lamps is the most widely used in electrical installations.

3.5.1.3 Series circuits versus parallel networks

Here are some points to remember:

1. In a series circuit, the total resistance is always greater than the greatest resistance in the circuit. This serves as a check when combining series resistances.
2. In a parallel network, the total resistance is always less than the smallest resistance in the network. Again this serves as a useful check especially as it is easy to forget to invert the term *1/R* during the evaluations of *R*.
3. To tell the difference between series and parallel, if in doubt imagine being an electron faced with the problem of passing through the circuits. If the electron has no choice but to pass through all the load units, then they are in series. If the electron has the choice of which load unit through which to pass, then the load units are in parallel.
4. Finally there are certain practical points to note about the two systems. For instance, with the parallel lamp arrangement, either lamp could fail to operate without affecting the operating of the other lamp. Lamp bulbs only last a certain length of time after which the filament breaks and the circuit is interrupted. This only interrupts the current flow in one branch and the remaining branch (or branches if there are more than two lamps) continues to pass current as before. However, if lamps are connected in series and one fails then all the lamps are extinguished. This is a problem that we sometimes see with Christmas tree lights. One bulb fails and all (or at least a number of) the lights go out. The advantage of series-connected bulbs is that they share the supply voltage, hence cheap low voltage lamps may be used. For most lighting purposes, this advantage is far outweighed by the unreliability of having most of the lamps giving out light should one fail and greater reliability is provided by the parallel

arrangement. Therefore the parallel arrangement is highly preferable in practical terms.

3.6 Revision Questions

1 The p.d's measured across three resistors connected in series are 5 V, 7 V and 10 V, and the supply current is 2 A. Determine (a) the supply voltage, (b) the total circuit resistance and (c) the values of the three resistors. [(a) 22 V (b) 11 Ω (c) 2.5 Ω, 3.5 Ω, 5 Ω]

2 For the circuit shown in Figure 5.26, determine the value of V_1. If the total circuit resistance is 36 Ω determine the supply current and the value of resistors R_1, R_2 and R_3.
[10 V, 0.5 A, 20 Ω, 10 Ω, 6 Ω]

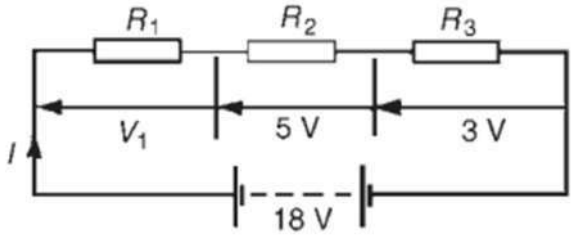


Figure 5.26

3 When the switch in the circuit in Figure 5.27 is closed the reading on voltmeter 1 is 30 V and that on voltmeter 2 is 10 V. Determine the reading on the ammeter and the value of resistor R_x
[4 A, 2.5 Ω]

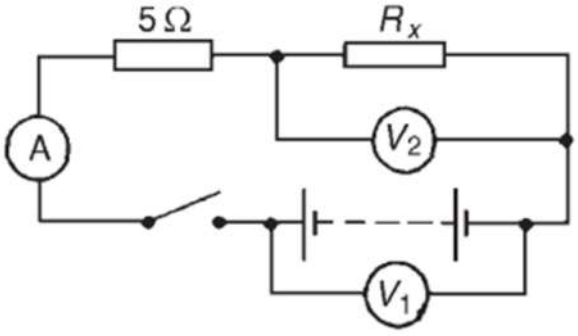


Figure 5.27

4 Two resistors are connected in series across an 18 V supply and a current of 5 A flows. If one of the resistors has a value of 2.4 Ω determine (a) the value of the other resistor and (b) the p.d. across the 2.4 Ω resistor. [(a) 1.2 Ω (b) 12 V]

5 Resistances of 4 Ω and 12 Ω are connected in parallel across a 9 V battery. Determine (a) the equivalent circuit resistance, (b) the supply current, and (c) the current in each resistor.
[(a) 3 Ω (b) 3 A (c) 2.25 A, 0.75 A]

6 For the circuit shown in Figure 5.28 determine (a) the reading on the ammeter, and (b) the value of resistor R. [2.5 A, 2.5 Ω]

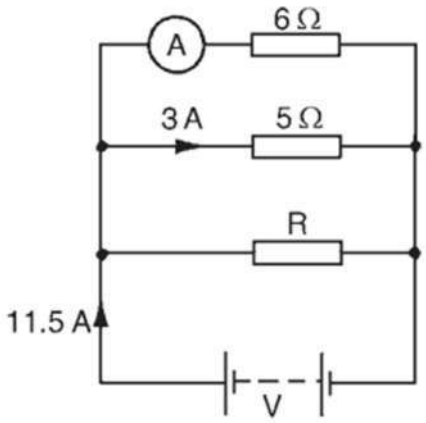


Figure 5.28

7 Find the equivalent resistance when the following resistances are connected (a) in series, (b) in parallel
(i) 3 Ω and 2 Ω (ii) 20 kΩ and 40 kΩ
(iii) 4 Ω, 8 Ω and 16 Ω (iv) 800 Ω, 4 kΩ and 1500 Ω
[(a) (i) 5 Ω (ii) 60 kΩ (iii) 28 Ω (iv) 6.3 kΩ
(b) (i) 1.2 Ω (ii) $13\frac{1}{3}$ kΩ (iii) $2\frac{2}{7}$ Ω (iv) 461.5 kΩ]

10 Resistors of 20 Ω, 20 Ω and 30 Ω are connected in parallel. What resistance must be added in series with the combination to obtain a total resistance of 10 Ω. If the complete circuit expends a power of 0.36 kW, find the total current flowing. [2.5 Ω, 6 A]

13 Find the current I in Figure 5.32. [1.8 A]

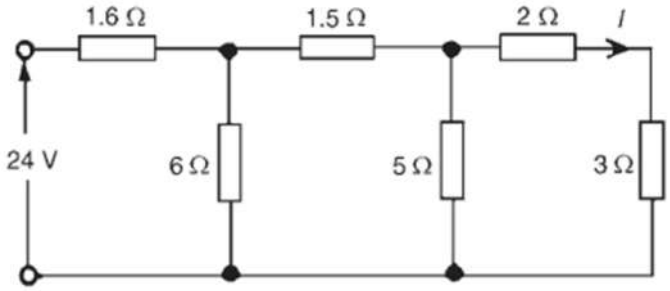


Figure 5.32

4 Capacitors and Capacitance

4.1 Electrostatic field

A field produced by an electric charge is called an **electric field**. That is, the space in which the effect of electric charge can be observed is termed as the electric field of the charge and the force on the charged body is the electric force. An **electrostatic field** is an electric field produced by a **stationary electric charge**.

The figure below represents two parallel metal plates, A and B, charged to different potentials.

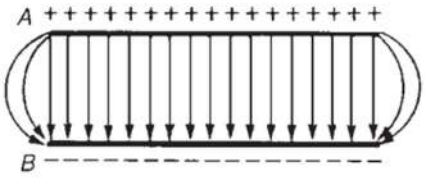


Figure 6.1 *Electrostatic field*

If an electron that has a negative charge is placed between the plates, a force will act on the electron tending to push it away from the negative plate B towards the positive plate, A. Similarly, a positive charge would be acted on by a force tending to move it toward the negative plate. Any region such as that shown between the plates above, in which an electric charge experiences a force, is called an **electrostatic field**. **The direction of the field is defined as that of the force acting on a positive charge placed in the field**.

Such a field may be represented in magnitude and direction by **lines of electric force** drawn between the charged surfaces. The **closeness** of the lines is an indication of the field strength. **A collection of the lines of force is known as flux (Ψ).**

Electric flux density (D) is the amount of flux passing through a defined area (A) that is perpendicular to the direction of the flux:

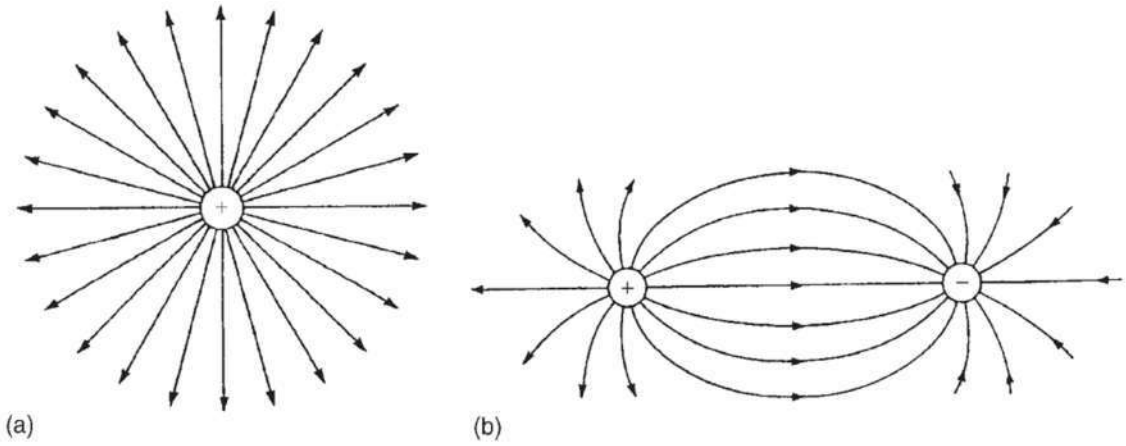


Figure 6.2 *(a) Isolated point charge; (b) adjacent charges of opposite polarity*

Electric lines of force (often called electric flux lines) are continuous and start and finish on point charges. Also, the lines cannot cross each other. When a charged body is placed close to an uncharged body, an induced charge of opposite sign appears on the surface of the uncharged body. This is because lines of force from the charged body terminate on its surface.

It should be noted that the space between the charged conductors needs to be filled with an insulator, otherwise the charges would move towards one another and therefore be dissipated. The insulant is called a dielectric.

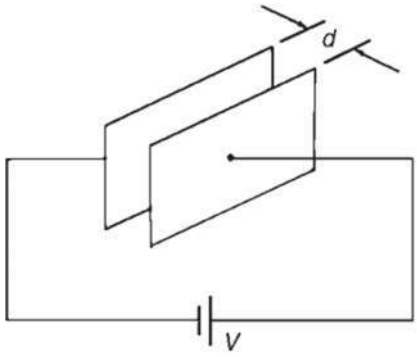
Coulomb's law

It states that "the **force of attraction or repulsion** between two electrically charged bodies is proportional to the magnitude of their charges and inversely proportional to the square of the distance separating them"

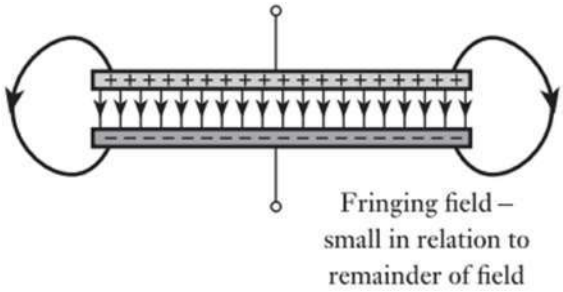
$$\text{i.e. force} \propto \frac{q_1 q_2}{d^2} \text{ or } \boxed{\text{force} = k\frac{q_1 q_2}{d^2}} \text{ where constant } k \approx 9 \times 10^9 \text{ in air}$$

4.2 Electric field strength

The figure below shows two parallel conducting plates separated from each other by air. They are connected to opposite terminals of a battery of voltage V volts.



There is therefore an electric field in the space between the plates. If the plates are close together, the electric lines of force will be straight and parallel and equally spaced, except near the edge where fringing will occur.



Over the area in which there is negligible fringing,

Electric field strength, $E = \frac{V}{d}$ volts/metre

where d is the distance between the plates. Electric field strength is also called **potential gradient**.

4.3 Capacitance

A device that is specifically designed to store electric charge is called a **capacitor / condenser**. In its simplest form a capacitor consists of two plates which are separated by an insulating material known as a **dielectric**.

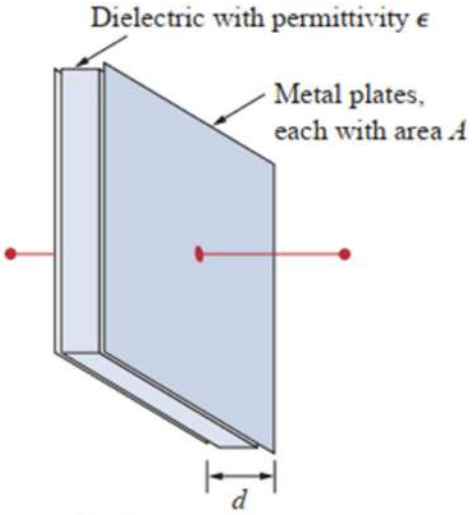


Figure 6.1
A typical capacitor.

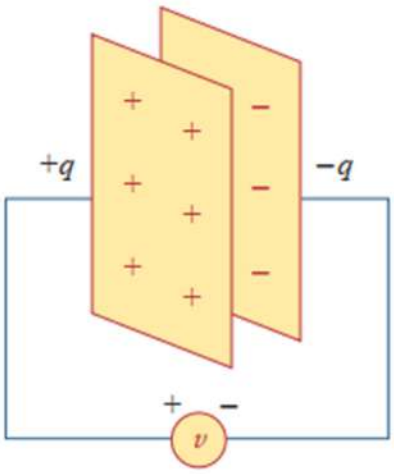


Figure 6.2
A capacitor with applied voltage v.

When a voltage source is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge *(+q)* on one plate and a negative charge *(–q)* on the other. The capacitor is said to store the electric charge. **The amount of charge stored, represented by q, is directly proportional to the applied voltage so that:**

$$q = Cv \qquad (6.1)$$

where C, the constant of proportionality, is known as the ***capacitance*** of the capacitor. Thus,

$$\text{capacitance } C = \frac{Q}{V}$$

The **unit of capacitance** is the **farad F** (or more usually $\mu F = 10^{-6}$ F or $pF = 10^{-12}$ F), which is defined as the capacitance when a pd of one volt appears across the plates when charged with one coulomb.

Although the capacitance *C* of a capacitor is the ratio of the charge *q* per plate to the applied voltage, it does not depend on *q* or *v*. It depends on the physical dimensions of the capacitor:

$$\text{Capacitance} \propto \frac{\text{area of plates}}{\text{distance between plates}}$$

For example, for the parallel-plate capacitor shown in Fig. 6.1, the capacitance is given by:

$$C = \frac{\epsilon A}{d}$$

where A is the surface area of each plate, d is the distance between the plates, and ϵ is the constant of proportionality **(permittivity** of the dielectric material between the plates). **Permittivity is the ability of a substance to store electric charge in an electric field. Capacitance can thus be defined as the ability of a capacitor to store electric charge.**

Although the above equation applies to only parallel-plate capacitors, we may infer from it that, in general, **three factors determine the value of the capacitance:**

1. The surface area of the plates - the larger the area, the greater the capacitance.
2. The spacing between the plates - the smaller the spacing, the greater the capacitance.
3. The permittivity of the material - the higher the permittivity, the greater the capacitance.

Capacitors are described by the dielectric material they are made of and by whether they are of fixed or variable type:

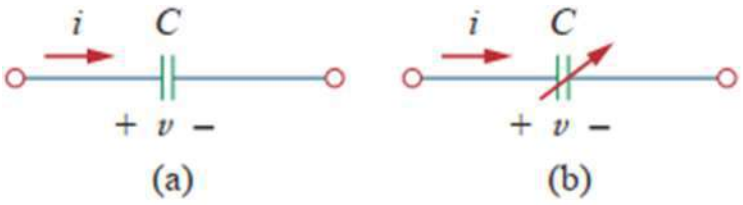


Figure 6.3
Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

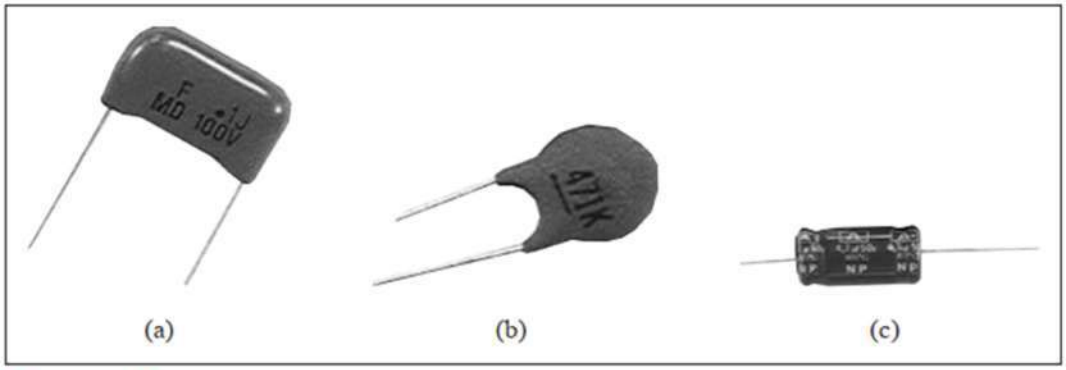


Figure 6.4
Fixed capacitors: (a) polyester capacitor, (b) ceramic capacitor, (c) electrolytic capacitor.
Courtesy of Tech America.

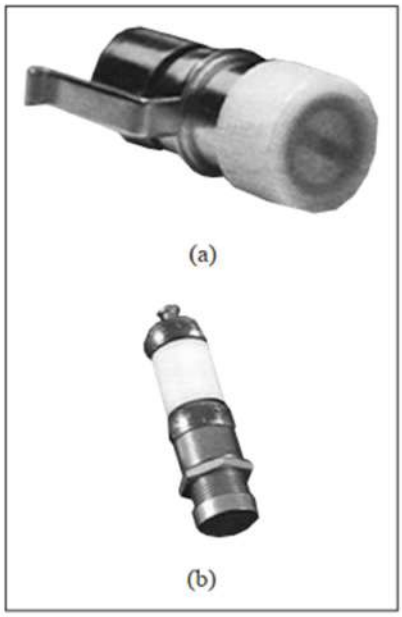


Figure 6.5
Variable capacitors: (a) trimmer capacitor, (b) filmtrim capacitor.
Courtesy of Johanson.

The capacitance of the variable air capacitor (meshed plates) is varied by turning the shaft. Variable capacitors are used in radio receivers allowing one to tune to various stations. In addition, capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise.

Every system of electrical conductors possesses capacitance. For example, there is capacitance between the conductors of overhead transmission lines and also between the wires of a telephone cable. In these examples the capacitance is undesirable but has to be accepted, minimized or compensated for. There are other situations where capacitance is a desirable property.

The **charge** Q stored in a capacitor is given by:

$$\boxed{Q = I \times t \text{ coulombs,}}$$

where I is the current in amperes and t the time in seconds.

For variable quantities:

$$i = \frac{dq}{dt}$$

and since $q = Cv$, differentiating both sides of the above equation gives:

$$\boxed{i = C\frac{dv}{dt}} \qquad (6.4)$$

This is the current-voltage relationship for a capacitor, assuming the passive sign convention. The relationship is illustrated in Fig. 6.6 for a capacitor whose capacitance is independent of voltage.

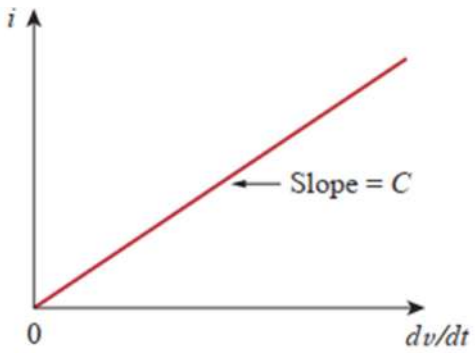


Figure 6.6
Current-voltage relationship of a capacitor.

According to Eq. (6.4), for a capacitor to carry current, its voltage must vary with time. Hence, for constant voltage, $i = 0$.

Capacitors that satisfy Eq. (6.4) are said to be *linear*. For a *nonlinear capacitor*, the plot of the current-voltage relationship is not a straight line. Although some capacitors are nonlinear, most are linear.

The voltage-current relation of the capacitor can be obtained by integrating both sides of Eq. (6.4). We get

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt \tag{6.5}$$

or

$$\boxed{v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)} \tag{6.6}$$

where $v(t_o)$ is the voltage across the capacitor at time t_0. Equation (6.6) shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory, a property that is often exploited.

The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt} \tag{6.7}$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{v(-\infty)}^{v(t)} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)} \tag{6.8}$$

We note that $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$. Thus,

$$\boxed{w = \frac{1}{2}Cv^2} \tag{6.9}$$

Using Eq. (6.1), we may rewrite Eq. (6.9) as

$$w = \frac{q^2}{2C} \tag{6.10}$$

Equation (6.9) or (6.10) represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy. In fact, the word *capacitor* is derived from this element's capacity to store energy in an electric field.

We should note the following important properties of a capacitor:

1. Note from Eq. (6.4) that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus, a capacitor is an open circuit to dc. However, if a battery (dc voltage) is connected across a capacitor, the capacitor charges.
2. The voltage on the capacitor must be continuous. The voltage on a capacitor cannot change abruptly. The capacitor resists an abrupt change in the voltage across it. According to Eq. (6.4), a discontinuous change in voltage requires an infinite current, which is physically impossible. For example, the voltage across a capacitor may take the form shown in Fig. 6.7(a), whereas it is not physically possible for the capacitor voltage to take the form shown in Fig. 6.7(b) because of the abrupt changes. Conversely, the current through a capacitor can change instantaneously.

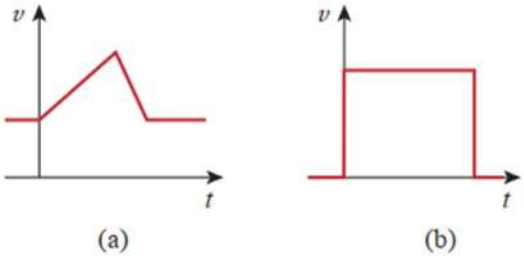


Figure 6.7
Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
4. A real, non-ideal capacitor has a parallel-model leakage resistance, as shown in Fig. 6.8. The leakage resistance may be as high as 100MΩ and can be neglected for most practical applications.

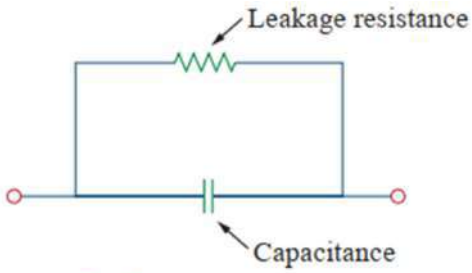


Figure 6.8
Circuit model of a nonideal capacitor.

Examples:

Problem 1. (a) Determine the p.d. across a 4 μF capacitor when charged with 5 mC.

(b) Find the charge on a 50 pF capacitor when the voltage applied to it is 2 kV.

(a) $C = 4\ \mu\text{F} = 4 \times 10^{-6}\ \text{F}$; $Q = 5\ \text{mC} = 5 \times 10^{-3}\ \text{C}$

Since $C = \dfrac{Q}{V}$ then $V = \dfrac{Q}{C} = \dfrac{5 \times 10^{-3}}{4 \times 10^{-6}} = \dfrac{5 \times 10^{6}}{4 \times 10^{3}} = \dfrac{5000}{4}$

Hence p.d. = 1250 V or 1.25 kV

(b) $C = 50\ \text{pF} = 50 \times 10^{-12}\ \text{F}$; $V = 2\ \text{kV} = 2000\ \text{V}$

$$Q = CV = 50 \times 10^{-12} \times 2000 = \frac{5 \times 2}{10^{8}} = 0.1 \times 10^{-6}$$

Hence charge = 0.1 μC

Problem 2. A direct current of 4 A flows into a previously uncharged 20 μF capacitor for 3 ms. Determine the pd between the plates.

$I = 4$ A; $C = 20\ \mu\text{F} = 20 \times 10^{-6}$ F; $t = 3$ ms $= 3 \times 10^{-3}$ s

$Q = It = 4 \times 3 \times 10^{-3}$ C

$$V = \frac{Q}{C} = \frac{4 \times 3 \times 10^{-3}}{20 \times 10^{-6}} = \frac{12 \times 10^{6}}{20 \times 10^{3}} = 0.6 \times 10^{3} = 600 \text{ V}$$

Hence, the pd between the plates is 600 V

Problem 3. A 5 μF capacitor is charged so that the pd between its plates is 800 V. Calculate how long the capacitor can provide an average discharge current of 2 mA.

$C = 5\ \mu\text{F} = 5 \times 10^{-6}$ F; $V = 800$ V; $I = 2$ mA $= 2 \times 10^{-3}$ A

$Q = CV = 5 \times 10^{-6} \times 800 = 4 \times 10^{-3}$ C

Also, $Q = It$. Thus, $t = \dfrac{Q}{I} = \dfrac{4 \times 10^{-3}}{2 \times 10^{-3}} = 2$ s

Hence the capacitor can provide an average discharge current of 2 mA for 2 s.

Example:

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.
(b) Find the energy stored in the capacitor.

Solution:

(a) Since $q = Cv$,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

(b) The energy stored is

$$w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

Example:

The voltage across a 5-μF capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

Solution:
By definition, the current is

$$i(t) = C\frac{dv}{dt} = 5 \times 10^{-6}\frac{d}{dt}(10 \cos 6000t)$$
$$= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A}$$

4.4 Electric flux density

Electric flux density, D, is the amount of flux passing through a defined area, A, which is perpendicular to the direction of the flux:

$$\textbf{electric flux density, } D = \frac{Q}{A} \textbf{ coulombs/metre}^2$$

Electric flux density is also called **charge density, σ**

4.5 Permittivity

Permittivity is the ability of a substance to store electric charge in an electric field.

At any point in an electric field, the electric field strength E maintains the electric flux and produces a particular value of electric flux density D at that point. For a field established in **vacuum** (or for practical purposes in air), the ratio D/E is a constant ε_0, i.e.

$$\boxed{\frac{D}{E} = \varepsilon_0}$$

where ε_0 is called the **permittivity of free space** or the free space constant. The value of ε_0 is 8.85×10^{-12} F/m.

When an insulating medium, such as mica, paper, plastic or ceramic, is introduced into the region of an electric field the ratio of D/E is modified:

$$\boxed{\frac{D}{E} = \varepsilon_0\varepsilon_r}$$

where ε_r, the **relative permittivity** of the insulating material, indicates its insulating power compared with that of vacuum:

$$\text{relative permittivity } \varepsilon_r = \frac{\text{flux density in material}}{\text{flux density in vacuum}}$$

ε_r has no unit. Typical values of ε_r include air, 1.00; polythene, 2.3; mica, 3–7; glass, 5–10; water, 80; ceramics, 6–1000.

The product $\varepsilon_0\varepsilon_r$ is called the **absolute permittivity**, ε, i.e.,

$$\boxed{\varepsilon = \varepsilon_0\varepsilon_r}$$

Problem 4. Two parallel rectangular plates measuring 20 cm by 40 cm carry an electric charge of 0.2 μC. Calculate the electric flux density. If the plates are spaced 5 mm apart and the voltage between them is 0.25 kV determine the electric field strength.

Charge $Q = 0.2\ \mu\text{C} = 0.2 \times 10^{-6} C$;

Area $A = 20\ \text{cm} \times 40\ \text{cm} = 800\ \text{cm}^2 = 800 \times 10^{-4}\ \text{m}^2$

Electric flux density $D = \dfrac{Q}{A} = \dfrac{0.2 \times 10^{-6}}{800 \times 10^{-4}} = \dfrac{0.2 \times 10^{4}}{800 \times 10^{6}}$

$$= \frac{2000}{800} \times 10^{-6} = \mathbf{2.5\ \mu C/m^2}$$

Voltage $V = 0.25\ \text{kV} = 250\ \text{V}$; Plate spacing, $d = 5\ \text{mm} = 5 \times 10^{-3}\ \text{m}$

Electric field strength $E = \dfrac{V}{d} = \dfrac{250}{5 \times 10^{-3}} = \mathbf{50\ kV/m}$

Problem 5. The flux density between two plates separated by mica of relative permittivity 5 is 2 μC/m^2. Find the voltage gradient between the plates.

Flux density $D = 2\ \mu C/m^2 = 2 \times 10^{-6}\ C/m^2$;

$\varepsilon_0 = 8.85 \times 10^{-12}$ F/m; $\varepsilon_r = 5$.

$\frac{D}{E} = \varepsilon_0\varepsilon_r$, hence **voltage gradient** $\boldsymbol{E} = \dfrac{D}{\varepsilon_0\varepsilon_r}$

$$= \frac{2 \times 10^{-6}}{8.85 \times 10^{-12} \times 5}\ \text{V/m}$$

$$= \mathbf{45.2\ kV/m}$$

Problem 6. Two parallel plates having a pd of 200 V between them are spaced 0.8 mm apart. What is the electric field strength? Find also the flux density when the dielectric between the plates is (a) air, and (b) polythene of relative permittivity 2.3

Electric field strength $\boldsymbol{E} = \dfrac{V}{D} = \dfrac{200}{0.8 \times 10^{-3}} = \mathbf{250\ kV/m}$

(a) For air: $\varepsilon_r = 1$

$\frac{D}{E} = \varepsilon_0\varepsilon_r$. Hence

electric flux density $\boldsymbol{D} = E\varepsilon_0\varepsilon_r$

$$= (250 \times 10^3 \times 8.85 \times 10^{-12} \times 1)\ C/m^2$$

$$= \mathbf{2.213\ \mu C/m^2}$$

(b) For polythene, $\varepsilon_r = 2.3$

Electric flux density $\boldsymbol{D} = E\varepsilon_0\varepsilon_r$

$$= (250 \times 10^3 \times 8.85 \times 10^{-12} \times 2.3)\ C/m^2$$

$$= \mathbf{5.089\ \mu C/m^2}$$

4.6 The parallel plate capacitor

Consider the figure below.

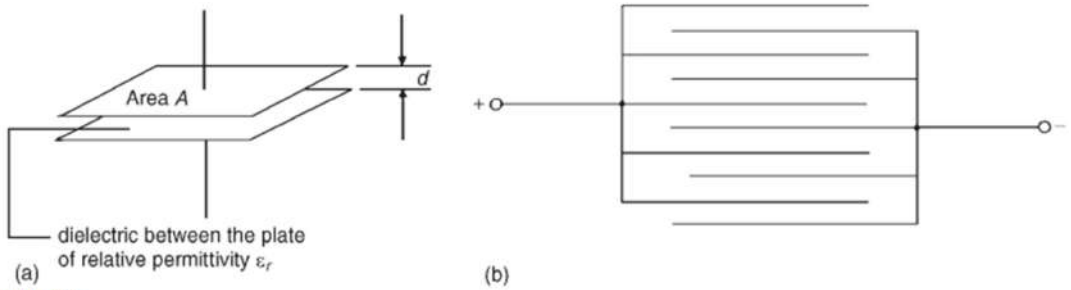


Figure 6.5

For a parallel-plate capacitor, as shown in Figure 6.5(a), experiments show that capacitance C is proportional to the area A of a plate, inversely proportional to the plate spacing d (i.e., the dielectric thickness) and depends on the nature of the dielectric:

$$C = \frac{\epsilon A}{d}$$

$$\textbf{Capacitance, } C = \frac{\varepsilon_0 \varepsilon_r A}{d} \textbf{ farads}$$

where $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m (constant)
ε_r = relative permittivity
A = area of one of the plates, in m^2, and
d = thickness of dielectric in m

One of the methods used to increase the capacitance of a capacitor is to interleave several plates as shown in Figure 6.5(b). **Ten plates are shown, forming nine capacitors** with a capacitance nine times that of one pair of plates. If such an arrangement has n plates then capacitance $C \propto (n-1)$.

Thus capacitance $$C = \frac{\varepsilon_0 \varepsilon_r A(n-1)}{d} \textbf{ farads}$$

Problem 8. A waxed paper capacitor has two parallel plates, each of effective area 800 cm^2. If the capacitance of the capacitor is 4425 pF determine the effective thickness of the paper if its relative permittivity is 2.5

$A = 800\ \text{cm}^2 = 800 \times 10^{-4}\ \text{m}^2 = 0.08\ \text{m}^2$;

$C = 4425\ \text{pF} = 4425 \times 10^{-12}\ \text{F}$; $\varepsilon_0 = 8.85 \times 10^{-12}\ \text{F/m}$; $\varepsilon_r = 2.5$

Since $C = \dfrac{\varepsilon_0 \varepsilon_r A}{d}$ then $d = \dfrac{\varepsilon_0 \varepsilon_r A}{C}$

Hence, $d = \dfrac{8.85 \times 10^{-12} \times 2.5 \times 0.08}{4425 \times 10^{-12}} = 0.0004\ \text{m}$

Hence the thickness of the paper is 0.4 mm

> Problem 9. A parallel plate capacitor has nineteen interleaved plates each 75 mm by 75 mm separated by mica sheets 0.2 mm thick. Assuming the relative permittivity of the mica is 5, calculate the capacitance of the capacitor.

$n = 19$; $n - 1 = 18$; $A = 75 \times 75 = 5625\ \text{mm}^2 = 5625 \times 10^{-6}\ \text{m}^2$;

$\varepsilon_r = 5$; $\varepsilon_0 = 8.85 \times 10^{-12}\ \text{F/m}$; $d = 0.2\ \text{mm} = 0.2 \times 10^{-3}\ \text{m}$

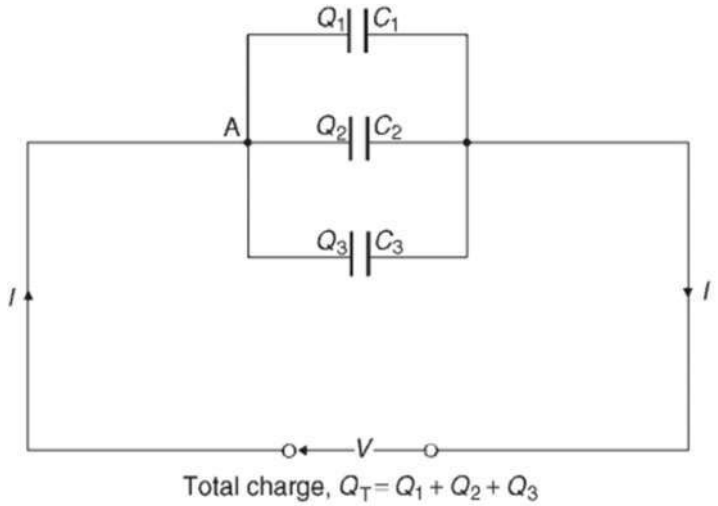
Capacitance $C = \dfrac{\varepsilon_0\ \varepsilon_r A(n-1)}{d}$

$$= \frac{8.85 \times 10^{-12} \times 5 \times 5625 \times 10^{-6} \times 18}{0.2 \times 10^{-3}}\ \text{F}$$

$$= \mathbf{0.0224\ \mu F\ or\ 22.4\ nF}$$

4.7 Capacitors connected in parallel

Consider the following circuit where three capacitors, C_1, C_2 and C_3, connected in parallel with a supply voltage V applied across the arrangement.



Total charge, $Q_T = Q_1 + Q_2 + Q_3$

When the charging current I reaches point A it divides, some flowing into C_1, some flowing into C_2 and some into C_3. Hence the total charge $Q_T (= I \times t)$ is divided between the three capacitors. The capacitors each store a charge and these are shown as Q_1, Q_2 and Q_3 respectively. Hence

$$Q_T = Q_1 + Q_2 + Q_3$$

But $Q_T = CV$, $Q_1 = C_1V$, $Q_2 = C_2V$ and $Q_3 = C_3V$
Therefore $CV = C_1V + C_2V + C_3V$ where C is the total equivalent circuit capacitance,

i.e. $\boldsymbol{C = C_1 + C_2 + C_3}$

It follows that for n parallel-connected capacitors,

$$\boxed{C = C_1 + C_2 + C_3 \ldots + C_n,}$$

The same can be obtained using calculus:

$$Q = it = Cv$$

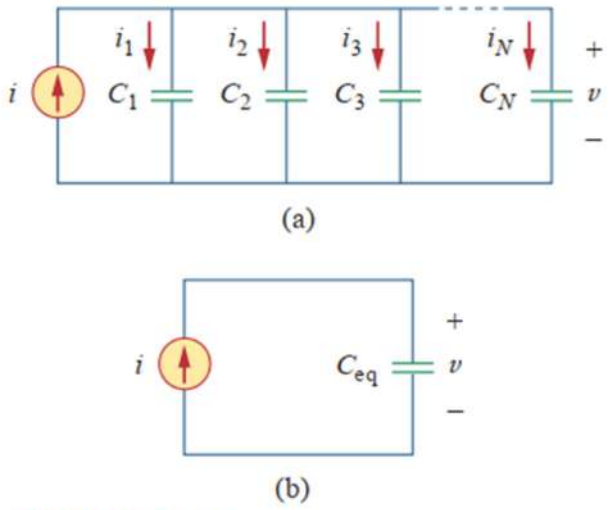


Figure 6.14
(a) Parallel-connected N capacitors,
(b) equivalent circuit for the parallel capacitors.

Applying KCL to the figure above;

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

But $i_k = C_k\, dv/dt$. Hence,

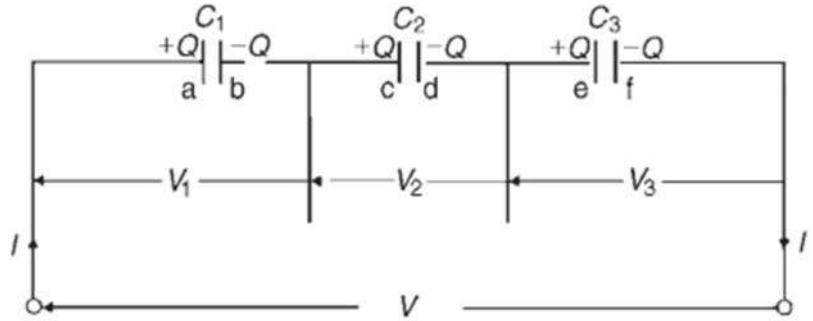
$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt}$$
$$= \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{eq}\frac{dv}{dt}$$

where

$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N$$

4.8 Capacitors connected in series

The figure below shows three capacitors, C_1, C_2 and C_3 connected in series across a supply voltage V. Let the p.d. across the individual capacitors be V_1, V_2 and V_3 respectively as shown.



Charge on each capacitor = Q

Since $Q = It$, when capacitors are connected in series the charge on each is the same.

Using KVL:

In a series circuit: $V = V_1 + V_2 + V_3$

Since $V = \frac{Q}{C}$ then $\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$

where C is the total equivalent circuit capacitance,

i.e. $\mathbf{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$

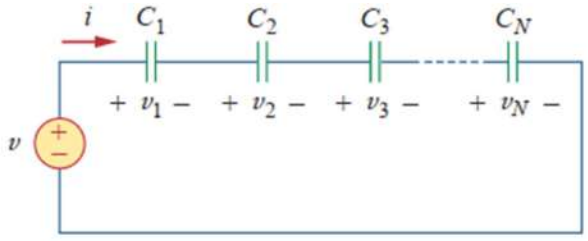
It follows that for n series-connected capacitors:

$$\boxed{\mathbf{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_n,}}}$$

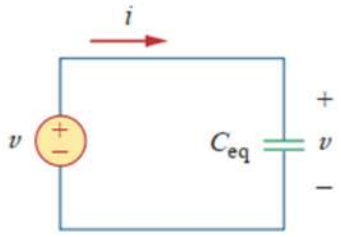
For the special case of **two capacitors in series:**

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{C_2 + C_1}{C_1\ C_2}$$

Hence $\boxed{\boldsymbol{C = \frac{C_1 C_2}{C_1 + C_2}}}$ $\left(\text{i.e. } \frac{\text{product}}{\text{sum}}\right)$



(a)



(b)

Figure 6.15
(a) Series-connected N capacitors,
(b) equivalent circuit for the series capacitor.

Example:

Find the equivalent capacitance seen between terminals a and b of the circuit in Fig. 6.16.

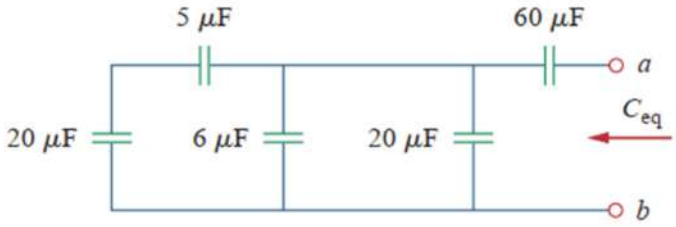


Figure 6.16
For Example 6.6.

Solution:
The 20-μF and 5-μF capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is

$$4 + 6 + 20 = 30\ \mu\text{F}$$

This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{eq} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

4.9 Dielectric strength

The maximum amount of field strength that a dielectric can withstand is called the dielectric strength of the material.

Dielectric strength, $\boxed{E_m = \frac{V_m}{d}}$

Problem 14. A capacitor is to be constructed so that its capacitance is 0.2 μF and to take a p.d. of 1.25 kV across its terminals. The dielectric is to be mica which, after allowing a safety factor of 2, has a dielectric strength of 50 MV/m. Find (a) the thickness of the mica needed, and (b) the area of a plate assuming a two-plate construction. (Assume ε_r for mica to be 6)

(a) Dielectric strength, $E = \frac{V}{d}$, i.e. $d = \frac{V}{E} = \frac{1.25 \times 10^3}{50 \times 10^6}$ m

$$= \mathbf{0.025\ mm}$$

(b) Capacitance, $C = \frac{\varepsilon_0 \varepsilon_r A}{d}$, hence area

$$A = \frac{Cd}{\varepsilon_0 \varepsilon_r} = \frac{0.2 \times 10^{-6} \times 0.025 \times 10^{-3}}{8.85 \times 10^{-12} \times 6}\ \text{m}^2$$

$$= 0.09416\ \text{m}^2 = \mathbf{941.6\ cm^2}$$

4.10 Charging and Discharging Capacitors

A Capacitor is a passive device that stores energy in its Electric Field and returns energy to the circuit whenever required. When a Capacitor is connected to a circuit with Direct Current (DC) source, two processes, which are called "charging" and "discharging" the Capacitor, will happen in specific conditions.

4.10.1 Charging Capacitors

Whenever we connect an uncharged or partially charged capacitor with a voltage source whose voltage is more than the voltage of the capacitor (in case of partially charged capacitor) it receives charge from the source and the voltage across the

capacitor rises exponentially until it becomes equal and opposite to voltage of the source.

Let us connect one capacitor of capacitance **C** in series with a resistor of resistance **R**. We also connect this series combination of capacitor and resistor with a battery of voltage **V** through a push switch **S**.

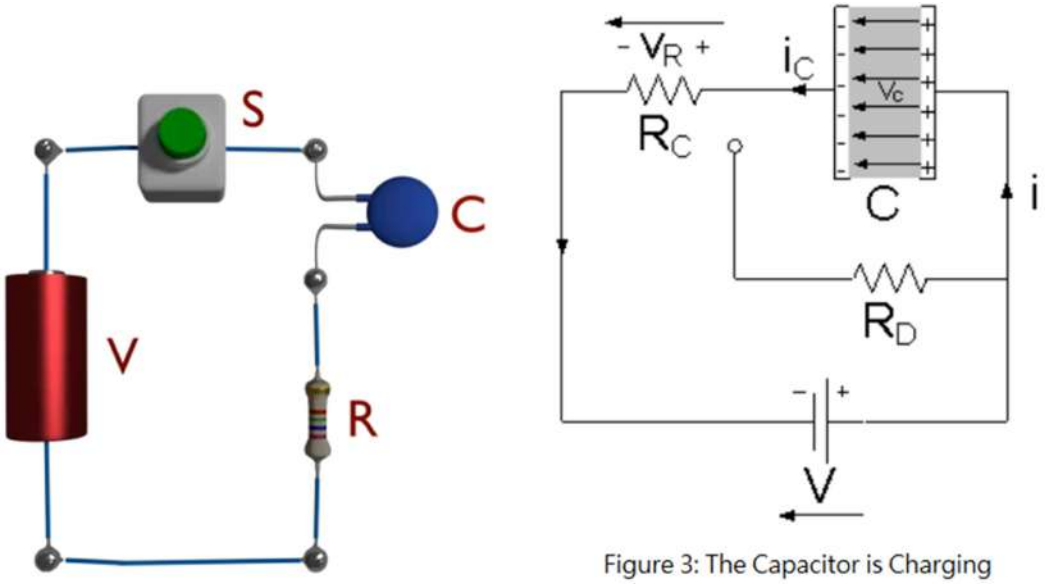


Figure 3: The Capacitor is Charging

Let us assume the capacitor is initially uncharged. When we push the switch, **as the capacitor is uncharged, no voltage gets developed across the capacitor, thus the capacitor will behave as a short circuit.** At that moment, the Capacitor is connected to the DC Power Supply and Current flows through the circuit and the charge just starts accumulating in the capacitor. **The current through the circuit will only be limited by resistance R (R_C).** Both Plates get the equal and opposite charges and an increasing Potential Difference, v_c, is created while the Capacitor is charging. Once the Voltage at the terminals of the Capacitor, v_c, is equal to the Power Supply Voltage, v_c = V, **the Capacitor is fully charged and the Current stops flowing through the circuit, the Charging Phase is over**.

Initially, the current through the circuit will only be limited by resistance R. So, the initial current is V/R. Now gradually the voltage is being developed across the capacitor, and this developed voltage is in the opposite of the polarity of the battery. As a result the current in the circuit gets gradually decreased. **When the voltage across the capacitor becomes equal**

and opposite of the voltage of the battery, the current becomes zero. During charging, the voltage across the capacitor gradually increases while the current through the capacitor gradually decreases.

In the derivations that follow, the voltage across the plates of a capacitor, v_c is taken as *v*.

The rate of increase of voltage across the capacitor is ***dv/dt*** at any instant ***t***. By definition, the current through the capacitor at that instant is

$$i = C\frac{dv}{dt}$$

Applying, Kirchhoff's Voltage Law, in the circuit at that instant, we can write,

$$V = v + Ri = v + RC\frac{dv}{dt} \quad \left[\because\ i = C\frac{dv}{dt}\right]$$

$$\Rightarrow V - v = RC\frac{dv}{dt} \Rightarrow \frac{dt}{CR} = \frac{dv}{V - v}$$

Integrating both side we get,

$$\int \frac{dt}{CR} = \int \frac{dv}{V - v} \Rightarrow \frac{t}{CR} = -\ ln(V - v) + A$$

Now, at the time of switching on the circuit, voltage across the capacitor was zero. That means, $v = 0$ **at** $t = 0$. Putting these values into the above equation, we get

$$\frac{0}{CR} = -\ ln(V - 0) + A$$
$$\Rightarrow 0 = -\ ln\, V + A$$
$$\Rightarrow A = ln\, V$$

After getting the value of A, we can rewrite the above equation as,

$$\frac{t}{CR} = -\ln(V - v) + \ln V$$
$$\Rightarrow \frac{t}{CR} = \ln\left(\frac{V}{V - v}\right)$$

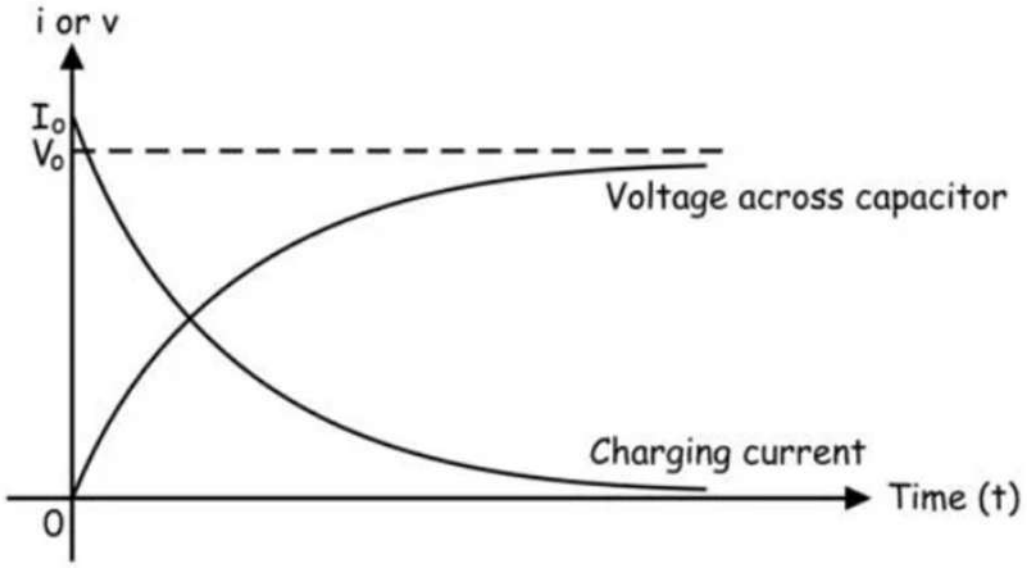
$$\Rightarrow \frac{V}{V - v} = e^{\frac{t}{CR}}$$

$$\Rightarrow V - v = Ve^{-\frac{t}{CR}}$$
$$\Rightarrow v = V(1 - e^{-\frac{t}{CR}})$$

Now, we know that,

$$i = C\frac{dv}{dt} = C\frac{d}{dt}\left[V(1 - e^{-\frac{t}{CR}})\right]$$
$$= CV\frac{d}{dt}(1 - e^{-\frac{t}{CR}}) = \frac{V}{R}e^{-\frac{t}{CR}}$$

This is the expression of charging current I, during the process of charging. The current and voltage of the capacitor during charging is shown below.



Here in the above figure, $\mathbf{I_o}$ is the initial current of the capacitor when it was initially uncharged during switching on the circuit and $\mathbf{V_o}$ is the final voltage after the capacitor gets fully charged.

Putting $\boldsymbol{t = RC}$ in the expression of charging current (as derived above), we get,

$$i = \frac{V}{R}\, e^{-1} = \frac{V}{R} \times 0.367$$

$$= 0.367\ I_o\ or\ 36.7\%\ of\ I_o$$

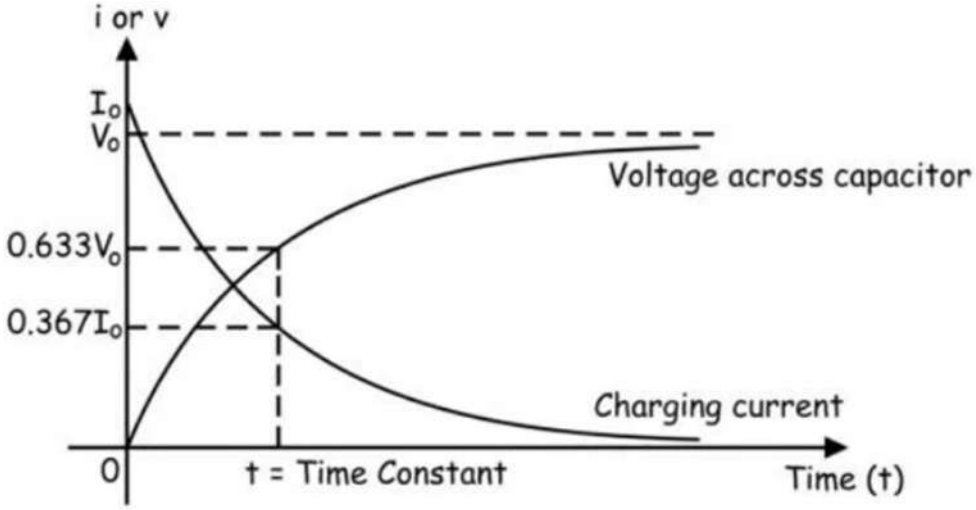
So at the time t = RC, the value of charging current becomes 36.7% of initial charging current ($V / R = I_o$) when the capacitor was fully uncharged. This time is known as the **time constant** of the capacitive circuit with capacitance value C farad along with the resistance R ohms in series with the capacitor. The value of voltage developed across that capacitor at the time constant is

$$v = V\ (1 - e^{-1}) = V \times (1 - 0.367)$$

$$= 0.633\ V\ or\ 63.3\%\ of\ V$$

$$= 63.3\%\ of\ V_o$$

Here $\mathbf{V_o}$ is the voltage finally developed across the capacitor after the capacitor is fully charged and it is same as source voltage ($\mathbf{V = V_o}$).



A fully charged Capacitor is equivalent to an Open-Circuit to Direct Current, R = ∞, because once the Charging Phase has finished, no more Current flows through it. The Voltage v_c on a Capacitor cannot change abruptly.

4.10.2 Discharging Capacitors

Discharging a capacitor means releasing the charge stored within the capacitor. A capacitor is discharged by connecting a high value resistor across the capacitor terminals.

We connect a charged capacitor of capacitance C farad in series with a resistor of resistance R ohms.

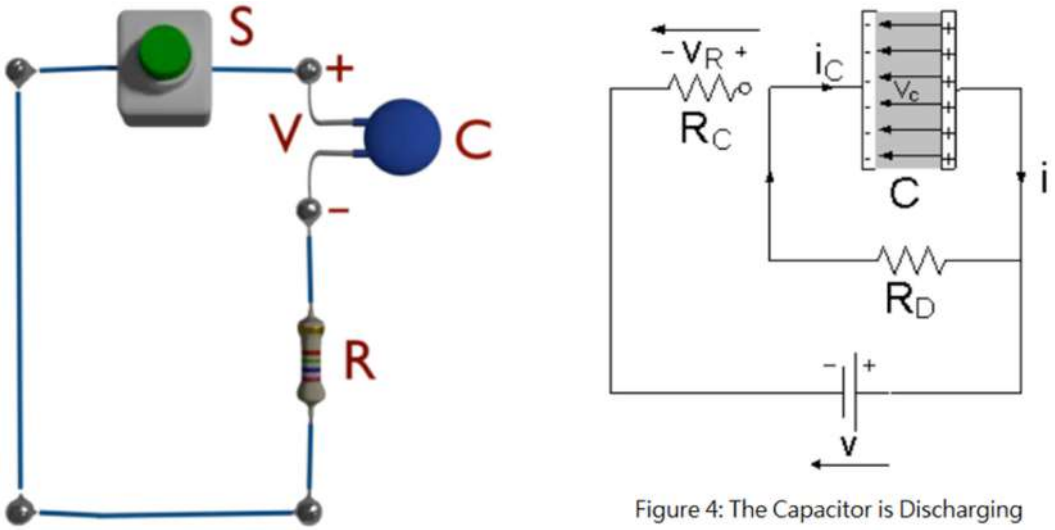


Figure 4: The Capacitor is Discharging

We then short circuit this series combination by switching on the push switch as shown.

As soon as the capacitor is short-circuited, it starts discharging.

Let us assume, the voltage of the capacitor at fully charged condition is V volts. As soon as the capacitor is short-circuited, the discharging current of the circuit would be – V / R ampere.

But after the instant of switching on that is at $t = {}^{+}0$, the current through the circuit is

$$i = C\frac{dv}{dt}$$

As per Kirchhoff's Voltage Law, we get,

$$v + iR = 0 \Rightarrow v + RC\frac{dv}{dt} = 0$$

$$\Rightarrow v = -RC\frac{dv}{dt} \Rightarrow \frac{dv}{v} = -\frac{dt}{RC}$$

Integrating both sides, we get,

$$\int \frac{dv}{v} = -\int \frac{dt}{RC} \Rightarrow ln\, v = -\frac{t}{RC} + A$$

Where, A is the constant of integration and, at **t = 0, v = V**,

$$ln\, V = -\frac{0}{RC} + A \Rightarrow A = ln\, V$$

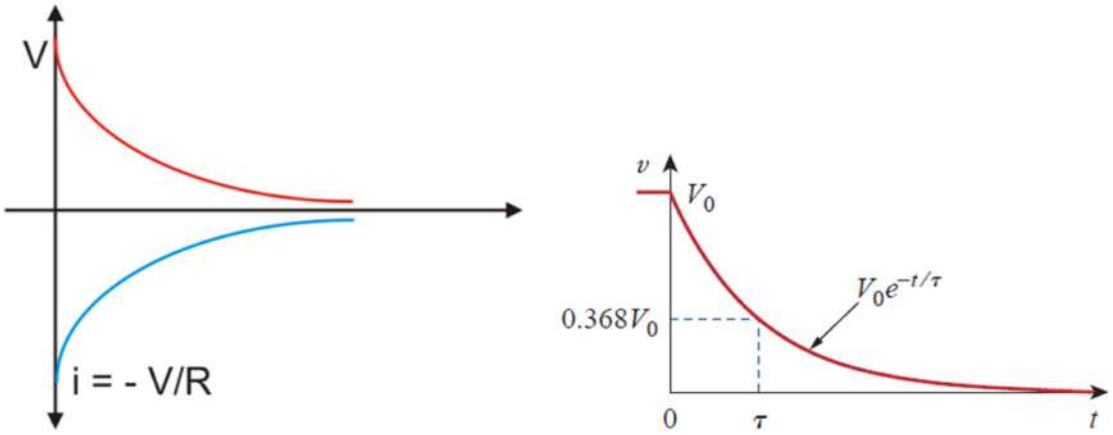
After calculating the value of A, we get,

$$Hence,\ ln\, v = -\frac{t}{RC} + ln\, V \Rightarrow ln\, \frac{v}{V} = -\frac{t}{RC}$$
$$\Rightarrow \frac{v}{V} = e^{-\frac{t}{RC}} \Rightarrow v = Ve^{-\frac{t}{RC}}$$

We know the form, KVL of the circuit,

$$i = -\frac{v}{R} = -\frac{1}{R}Ve^{-\frac{t}{RC}}$$
$$= -\frac{V}{R}e^{-\frac{t}{RC}} = -Ie^{-\frac{t}{RC}} \left[Here,\ I = \frac{V}{R}\right]$$

If we plot these discharging current and voltage in the graph, we get,



Hence the capacitor current exponentially reaches zero from its initial value, and the capacitor voltage reaches exponentially to zero from its initial value during discharging.

The rapidity with which the voltage / current decreases is expressed in terms of the ***time constant***, denoted by τ, the lowercase Greek letter tau. The time constant of a circuit is the time required for the response to decay to a factor of 1 / e or 36.8 percent of its initial value.

With a calculator it is easy to verify the table below:

Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

It is evident from the table above that the voltage $v(t)$ is less than 1% of V_0 after 5τ (five time constants). Thus, it is usual to assume that the capacitor is fully discharged (or charged) after five time constants. That is, it takes five time constants for the circuit to reach its final state or steady state when no changes take place with time.

Note that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated below.

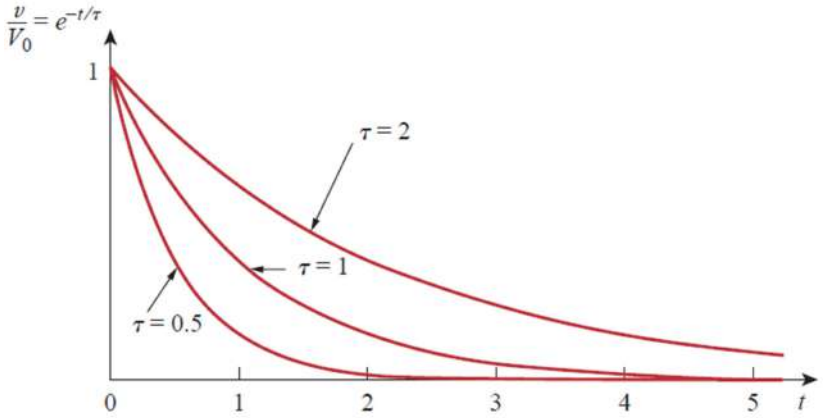


Figure 7.4
Plot of $v/V_0 = e^{-t/\tau}$ for various values of the time constant.

4.11 Revision Questions

Charge and capacitance

1 Find the charge on a 10 μF capacitor when the applied voltage is 250 V. [2.5 mC]

2 Determine the voltage across a 1000 pF capacitor to charge it with 2 μC. [2 kV]

3 The charge on the plates of a capacitor is 6 mC when the potential between them is 2.4 kV. Determine the capacitance of the capacitor. [2.5 μF]

4 For how long must a charging current of 2 A be fed to a 5 μF capacitor to raise the pd between its plates by 500 V. [1.25 ms]

5 A steady current of 10 A flows into a previously uncharged capacitor for 1.5 ms when the pd between the plates is 2 kV. Find the capacitance of the capacitor. [7.5 μF]

Electric field strength, electric flux density and permittivity

6 A capacitor uses a dielectric 0.04 mm thick and operates at 30 V. What is the electric field strength across the dielectric at this voltage? [750 kV/m]

7 A two-plate capacitor has a charge of 25 C. If the effective area of each plate is 5 cm^2 find the electric flux density of the electric field. [50 kC/m^2]

8 A charge of 1.5 μC is carried on two parallel rectangular plates each measuring 60 mm by 80 mm. Calculate the electric flux density. If the plates are spaced 10 mm apart and the voltage between them is 0.5 kV determine the electric field strength. [312.5 $\mu C/m^2$, 50 kV/m]

9 The electric flux density between two plates separated by polystyrene of relative permittivity 2.5 is 5 $\mu C/m^2$. Find the voltage gradient between the plates. [226 kV/m]

10 Two parallel plates having a pd of 250 V between them are spaced 1 mm apart. Determine the electric field strength. Find also the electric flux density when the dielectric between the plates is (a) air and (b) mica of relative permittivity 5. [250 kV/m (a) 2.213 $\mu C/m^2$ (b) 11.063 $\mu C/m^2$]

Parallel plate capacitor

11 A capacitor consists of two parallel plates each of area 0.01 m^2, spaced 0.1 mm in air. Calculate the capacitance in picofarads. [885 pF]

12 A waxed paper capacitor has two parallel plates, each of effective area 0.2 m^2. If the capacitance is 4000 pF determine the effective thickness of the paper if its relative permittivity is 2. [0.885 mm]

13 Calculate the capacitance of a parallel plate capacitor having 5 plates, each 30 mm by 20 mm and separated by a dielectric 0.75 mm thick having a relative permittivity of 2.3 [65.14 pF]

14 How many plates has a parallel plate capacitor having a capacitance of 5 nF, if each plate is 40 mm by 40 mm and each dielectric is 0.102 mm thick with a relative permittivity of 6. [7]

15 A parallel plate capacitor is made from 25 plates, each 70 mm by 120 mm interleaved with mica of relative permittivity 5. If the capacitance of the capacitor is 3000 pF determine the thickness of the mica sheet. [2.97 mm]

16 The capacitance of a parallel plate capacitor is 1000 pF. It has 19 plates, each 50 mm by 30 mm separated by a dielectric of thickness 0.40 mm. Determine the relative permittivity of the dielectric. [1.67]

17 A capacitor is to be constructed so that its capacitance is 4250 pF and to operate at a pd of 100 V across its terminals. The dielectric is to be polythene ($\varepsilon_r = 2.3$) which, after allowing a safety factor, has a dielectric strength of 20 MV/m. Find (a) the thickness of the polythene needed, and (b) the area of a plate. [(a) 0.005 mm (b) 10.44 cm^2]

Capacitors in parallel and series

18 Capacitors of 2 μF and 6 μF are connected (a) in parallel and (b) in series. Determine the equivalent capacitance in each case. [(a) 8 μF (b) 1.5 μF]

19 Find the capacitance to be connected in series with a 10 μF capacitor for the equivalent capacitance to be 6 μF [15 μF]

20 Two 6 μF capacitors are connected in series with one having a capacitance of 12 μF. Find the total equivalent circuit capacitance. What capacitance must be added in series to obtain a capacitance of 1.2 μF? [2.4 μF, 2.4 μF]

22 For the arrangement shown in Figure 6.15 find (a) the equivalent circuit capacitance and (b) the voltage across a 4.5 μF capacitor.
[(a) 1.2 μF (b) 100 V]

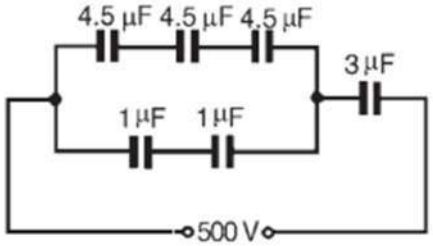


Figure 6.15

Practice Problem 6.6

Find the equivalent capacitance seen at the terminals of the circuit in Fig. 6.17.

Answer: 40 μF.

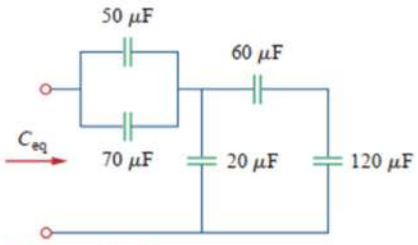


Figure 6.17
For Practice Prob. 6.6.

23 Three 12 μF capacitors are connected in series across a 750 V supply. Calculate (a) the equivalent capacitance, (b) the charge on each capacitor and (c) the pd across each capacitor.
[(a) 4 μF (b) 3 mC (c) 250 V]

24 If two capacitors having capacitances of 3 μF and 5 μF respectively are connected in series across a 240 V supply, determine (a) the p.d. across each capacitor and (b) the charge on each capacitor.
[(a) 150 V, 90 V (b) 0.45 mC on each]

25 In Figure 6.16 capacitors P, Q and R are identical and the total equivalent capacitance of the circuit is 3 μF. Determine the values of P, Q and R [4.2 μF each]

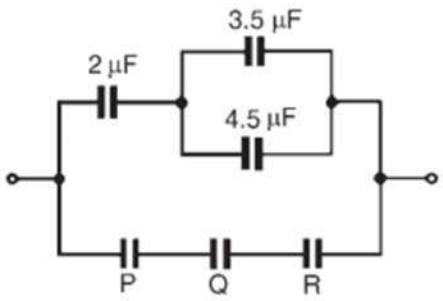


Figure 6.16

Energy stored

26 When a capacitor is connected across a 200 V supply the charge is 4 μC. Find (a) the capacitance and (b) the energy stored.
[(a) 0.02 μF (b) 0.4 mJ]

27 Find the energy stored in a 10 μF capacitor when charged to 2 kV.
[20 J]

28 A 3300 pF capacitor is required to store 0.5 mJ of energy. Find the pd to which the capacitor must be charged. [550 V]

29 A capacitor, consisting of two metal plates each of area 50 cm^2 and spaced 0.2 mm apart in air, is connected across a 120 V supply. Calculate (a) the energy stored, (b) the electric flux density and (c) the potential gradient
[(a) 1.593 μJ (b) 5.31 $\mu C/m^2$ (c) 600 kV/m]

30 A bakelite capacitor is to be constructed to have a capacitance of 0.04 μF and to have a steady working potential of 1 kV maximum. Allowing a safe value of field stress of 25 MV/m find (a) the thickness of bakelite required, (b) the area of plate required if the relative permittivity of bakelite is 5, (c) the maximum energy stored by the capacitor and (d) the average power developed if this energy is dissipated in a time of 20 μs.
[(a) 0.04 mm (b) 361.6 cm^2 (c) 0.02 J (d) 1 kW]

5 Inductors (*also coil or choke*)

An inductor is a passive element designed **to store energy in its magnetic field**. Any conductor of electric current has inductive properties and may be regarded as an inductor. But in order to **enhance the inductive effect**, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown below:

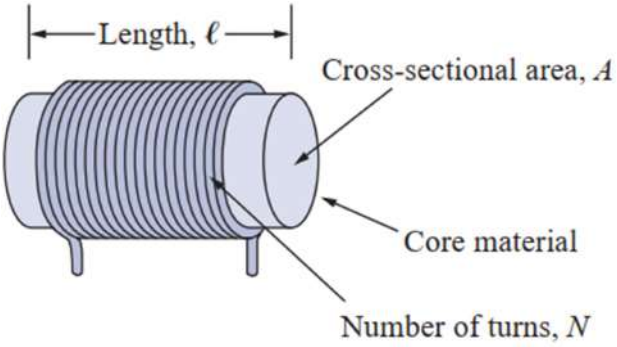


Figure 6.21
Typical form of an inductor.

If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$v = L\frac{di}{dt} \qquad (6.18)$$

where L is the constant of proportionality called the ***inductance*** of the inductor. The unit of inductance is the henry (H). **Inductance** is the property whereby an inductor exhibits opposition to the change of current flowing through it.

Equation (6.18) is the voltage-current relationship for an inductor. Figure 6.24 shows this relationship graphically for an inductor whose inductance is independent of current. Such an inductor is known as a ***linear inductor***. For a ***nonlinear inductor***, the plot of Eq. (6.18) will not be a straight line because its inductance varies with current.

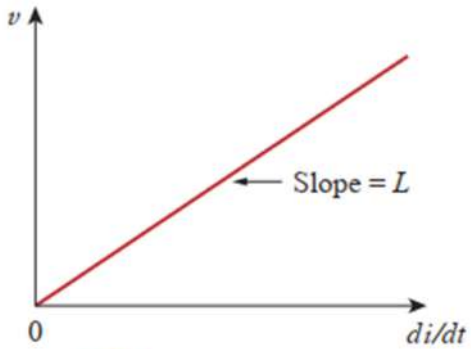


Figure 6.24
Voltage-current relationship of an inductor.

The inductance of an inductor depends on its physical dimension and construction. Consequently, formulas for calculating the inductance of inductors of different shapes are different. For example, for the inductor, (solenoid) shown in Fig. 6.21,

$$L = \frac{N^2 \mu A}{\ell} \qquad (6.19)$$

where N is the number of turns, l is the length, A is the cross-sectional area, and μ is the **permeability** of the core. We can see from Eq. (6.19) that inductance can be increased by increasing the number of turns of coil, using material with higher permeability as the core, increasing the cross-sectional area, or reducing the length of the coil.

Inductors may be fixed or variable. The core may be made of iron, steel, plastic, or air. Common inductors are shown in Fig. 6.22. The circuit symbols for inductors are shown in Fig. 6.23, following the passive sign convention.

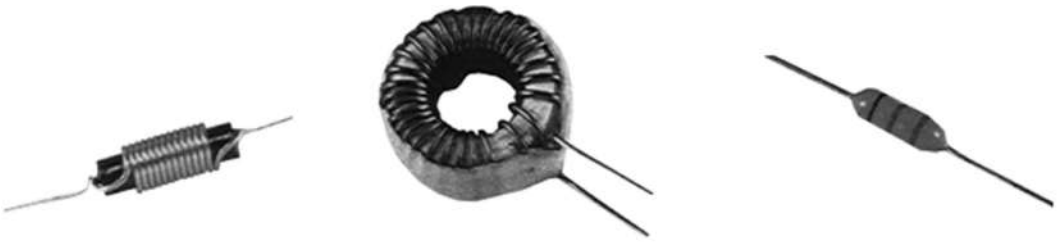


Figure 6.22
Various types of inductors: (a) solenoidal wound inductor, (b) toroidal inductor, (c) chip inductor.
Courtesy of Tech America.

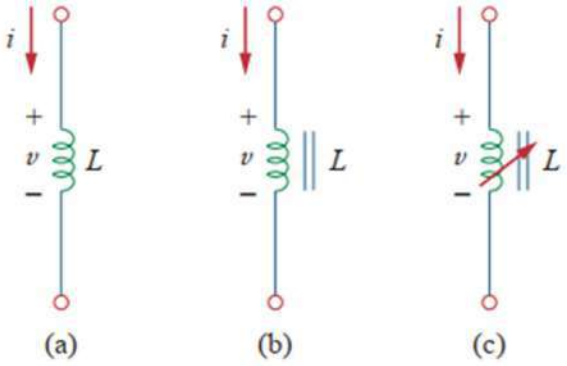


Figure 6.23
Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.

The current-voltage relationship is obtained from Eq. (6.18) as

$$di = \frac{1}{L} v\, dt$$

Integrating gives

$$i = \frac{1}{L} \int_{-\infty}^{t} v(t)\, dt \qquad \textbf{(6.20)}$$

or

$$\boxed{i = \frac{1}{L} \int_{t_0}^{t} v(t)\, dt + i(t_0)} \qquad \textbf{(6.21)}$$

where $i(t_0)$ is the total current for $-\infty < t < t_0$ and $i(-\infty) = 0$. The idea of making $i(-\infty) = 0$ is practical and reasonable, because there must be a time in the past when there was no current in the inductor.

The inductor is designed to store energy in its magnetic field. The energy stored can be obtained from Eq. (6.18). The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i \tag{6.22}$$

The energy stored is

$$\begin{aligned} w &= \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} \left(L\frac{di}{dt}\right)i\,dt \\ &= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty) \end{aligned} \tag{6.23}$$

Since $i(-\infty) = 0$,

$$\boxed{w = \frac{1}{2}Li^2} \tag{6.24}$$

5.1 Important properties of an inductor

1. Note from Eq. (6.18) that the voltage across an inductor is zero when the current is constant. **Thus, an inductor acts like a short circuit to dc**.
2. An important property of the inductor is its opposition to the change in current flowing through it. **The current through an inductor cannot change instantaneously (abruptly).** However, the voltage across an inductor can change abruptly.

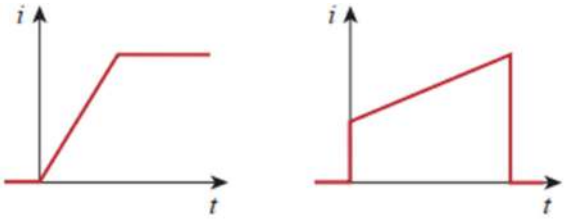


Figure 6.25
Current through an inductor: (a) allowed, (b) not allowable; an abrupt change is not possible.

3. Like the ideal capacitor, **the ideal inductor does not dissipate energy**. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.
4. An **ideal inductor has no resistance**. However, a practical, non-ideal inductor has a significant resistive component, as shown in Fig. 6.26. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the ***winding resistance*, R_w**, and it appears in series with the inductance of the inductor. The presence of R_w makes it both an energy storage device and an energy dissipation device. Since R_w is usually very small, it is ignored in most cases. The non-ideal inductor also has a ***winding capacitance*** C_w due to the capacitive coupling between the conducting coils. *Cw* is very small and can be ignored in most cases, except at high frequencies.

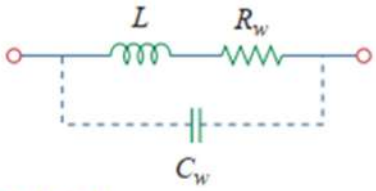


Figure 6.26
Circuit model for a practical inductor.

Since an inductor is often made of a highly conducting wire, it has a very small resistance.

Example:

The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

Solution:

Since $v = L\,di/dt$ and $L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1 - 5t)\text{ V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\text{ J}$$

5.2 Inductors connected in Series

Consider a series connection of N inductors, as shown in Fig. 6.29(a), with the equivalent circuit shown in Fig. 6.29(b). The inductors have the same current through them.

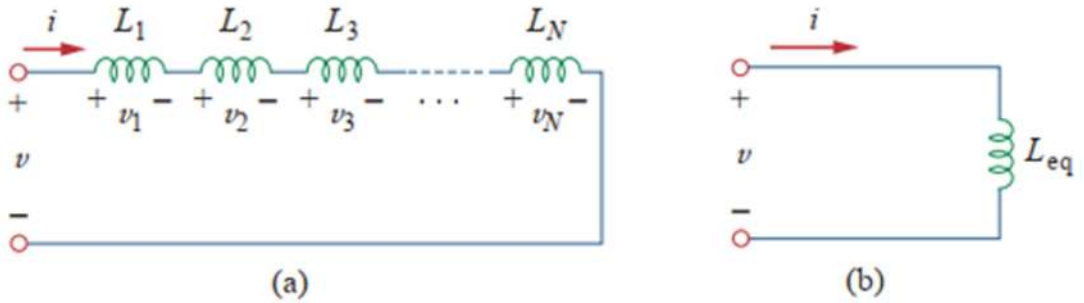


Figure 6.29
(a) A series connection of N inductors,
(b) equivalent circuit for the series inductors.

Applying **KVL** to the loop,

$$v = v_1 + v_2 + v_3 + \cdots + v_N \qquad \textbf{(6.25)}$$

Substituting $v_k = L_k\, di/dt$ results in

$$v = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \cdots + L_N\frac{di}{dt}$$

$$= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt} \qquad \textbf{(6.26)}$$

$$= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{eq}\frac{di}{dt}$$

where

$$\boxed{L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N} \qquad \textbf{(6.27)}$$

Thus, the equivalent inductance of series-connected inductors is the sum of the individual inductances (**Inductors in series are combined in exactly the same way as resistors in series**).

5.3 Inductors connected in Parallel

Consider a parallel connection of *N* inductors, as shown in Fig. 6.30(a), with the equivalent circuit in Fig. 6.30(b).

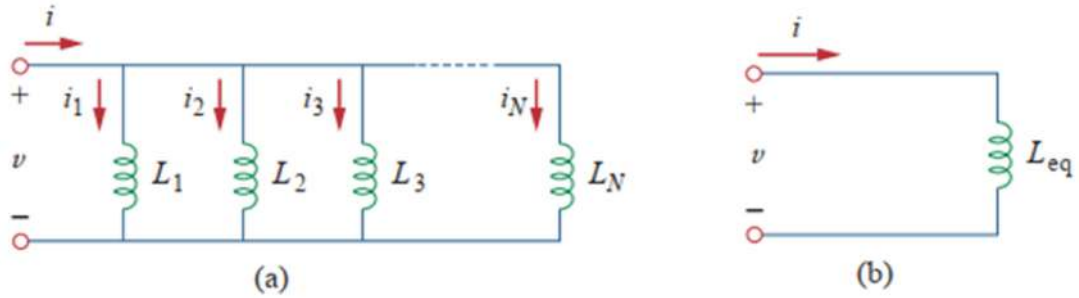


Figure 6.30
(a) A parallel connection of *N* inductors,
(b) equivalent circuit for the parallel inductors.

The inductors have the same voltage across them. Using **KCL**,

$$i = i_1 + i_2 + i_3 + \cdots + i_N \tag{6.28}$$

But $i_k = \dfrac{1}{L_k}\int_{t_0}^{t} v\,dt + i_k(t_0)$; hence,

$$\begin{aligned} i &= \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) \\ &\quad + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0) \\ &= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0) \\ &\quad + \cdots + i_N(t_0) \\ &= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{\text{eq}}}\int_{t_0}^{t} v\,dt + i(t_0) \end{aligned} \tag{6.29}$$

where

$$\boxed{\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}} \tag{6.30}$$

Thus, the equivalent inductance of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances (**the inductors in parallel are combined in the same way as resistors in parallel**).

The initial current $i(t_0)$ through L_{eq} at $t = t_0$ is expected by KCL to be the sum of the inductor currents at t_0. Thus, according to Eq. (6.29),

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

For two inductors in parallel, Eq. (6.30) becomes

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} \quad \text{or} \quad L_{\text{eq}} = \frac{L_1 L_2}{L_1 + L_2} \tag{6.31}$$

TABLE 6.1

Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

† *Passive sign convention is assumed.*

Example 6.11

Find the equivalent inductance of the circuit shown in Fig. 6.31.

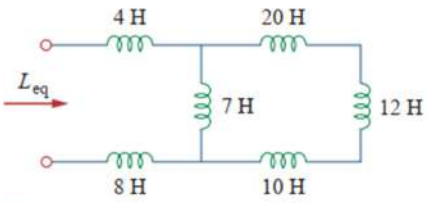


Figure 6.31
For Example 6.11.

Solution:

The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

$$\frac{7 \times 42}{7 + 42} = 6\ \text{H}$$

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence,

$$L_{eq} = 4 + 6 + 8 = 18\ \text{H}$$

Example 6.12

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

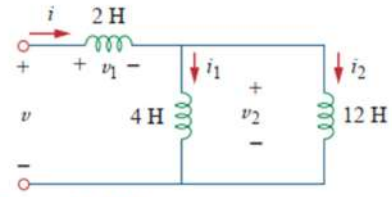


Figure 6.33
For Example 6.12.

Solution:

(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \parallel 12 = 2 + 3 = 5 \text{ H}$$

Thus,

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5 \text{ mA}$$
$$= -3e^{-10t}\Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA}$$

Similarly,

$$i_2(t) = \frac{1}{12}\int_0^t v_2\,dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\,dt - 1 \text{ mA}$$
$$= -e^{-10t}\Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA}$$

Note that $i_1(t) + i_2(t) = i(t)$.

5.4 Applications

Unlike capacitors and resistors, inductors with appreciable inductance are difficult to produce on IC substrates. Therefore, inductors (coils) usually come in discrete form and tend to be

more bulky and expensive. For this reason, inductors are not as versatile as capacitors and resistors, and they are more limited in applications. However, there are several applications in which inductors have no practical substitute. They are routinely used in relays, delays, sensing devices, telephone circuits, radio and TV receivers, power supplies, electric motors, microphones, and loudspeakers, to mention a few.

5.5 Conclusion

Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

1. The capacity to store energy makes them useful as **temporary voltage or current sources**. Thus, they can be used for generating a large amount of current or voltage for a short period of time.
2. Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property makes inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage.
3. Capacitors and inductors are frequency sensitive. This property makes them useful for **frequency discrimination**.

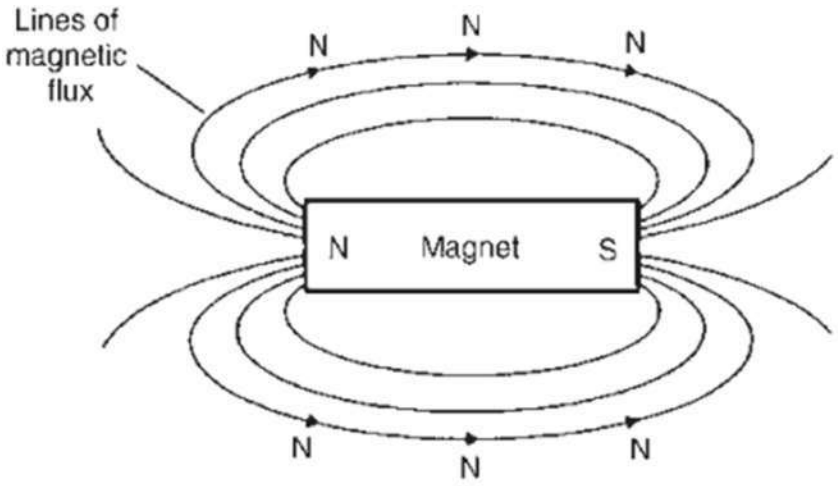
The first two properties are put to use in dc circuits, while the third one is taken advantage of in ac circuits.

6 Magnetic circuits

6.1 Introduction

The north-seeking end of the magnet is called the **North Pole**, ***N***, and the south-seeking end the **South Pole**, ***S***. The area around a magnet is called the **magnetic field** and it is in this area that the effects of the **magnetic force** produced by the magnet can be detected.

The concept of **lines of magnetic flux was put forward by Michael Faraday** as a means of visualizing the distribution and density of a magnetic field. These lines of magnetic flux have no physical existence.



6.2 Characteristics of lines of magnetic flux

Consider the figures given below:

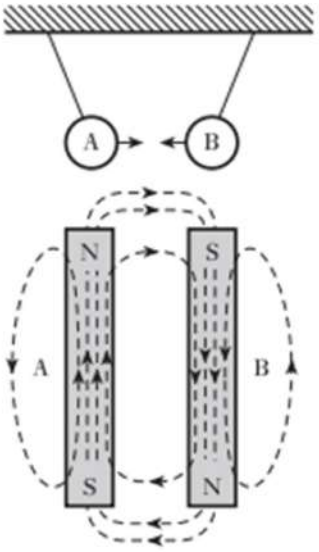


Fig. 6.4 Attraction between magnets

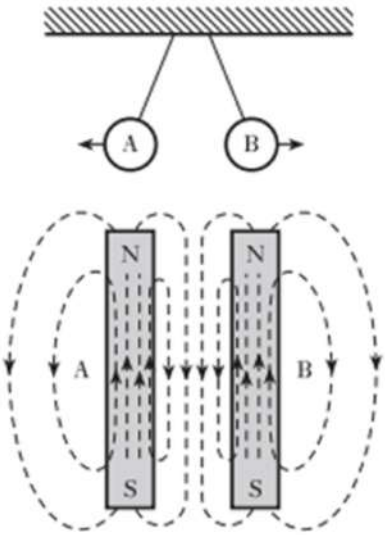


Fig. 6.5 Repulsion between magnets

1. *The direction of a line of magnetic flux at any point in a non-magnetic medium, such as air, is that of the north-seeking pole of a compass needle placed at that point*.
2. *Each line of magnetic flux forms a closed loop*, as shown by the dotted lines in Figs 6.4 and 6.5. This means that a line of flux emerging from any point at the N-pole end of a magnet passes through the surrounding space back to the S-pole end and is then assumed to continue through the magnet to the point at which it emerged at the N-pole end.
3. *Lines of magnetic flux never intersect.* This follows from the fact that if a compass needle is placed in a magnetic field, its north-seeking pole will point in one direction only, namely in the direction of the magnetic flux at that point.
4. *Lines of magnetic flux are like stretched elastic cords, always trying to shorten themselves.* This effect can be demonstrated by suspending two permanent magnets, A and B, parallel to each other, with their poles arranged as in Fig. 6.4. The distribution of the resultant magnetic field is indicated by the dotted lines. The lines of magnetic flux passing between A and B behave as if they were in tension, trying to shorten themselves and thereby causing the magnets to be attracted towards each other. In other words, unlike poles attract each other.
5. *Lines of magnetic flux which are parallel and in the same direction repel one another*. This effect can be demonstrated by suspending the two permanent magnets, A and B,

with their N poles pointing in the same direction, as in Fig. 6.5. It will be seen that in the space between A and B the lines of flux are practically parallel and are in the same direction. These flux lines behave as if they exerted a lateral pressure on one another, thereby causing magnets A and B to repel each other. Hence like poles repel each other.

6.3 Magnetic flux and flux density

Magnetic flux is the amount of magnetic field (or the number of lines of force) produced by a magnetic source. The symbol for magnetic flux is Φ (Greek letter 'phi'). The unit of magnetic flux is the **weber**, **Wb.** Magnetic flux density is the amount of flux passing through a defined area that is perpendicular to the direction of the flux:

$$\textbf{Magnetic flux density} = \frac{\textbf{magnetic flux}}{\textbf{area}}$$

The symbol for magnetic flux density is **B**. The unit of magnetic flux density is the **Tesla, T.**

$$\boxed{B = \frac{\Phi}{A}\ \textbf{tesla}}, \text{ where } A(\text{m}^2) \text{ is the area}$$

Problem 2. The maximum working flux density of a lifting electromagnet is 1.8 T and the effective area of a pole face is circular in cross-section. If the total magnetic flux produced is 353 mWb, determine the radius of the pole face.

Flux density $B = 1.8$ T; flux $\Phi = 353$ mWb $= 353 \times 10^{-3}$ Wb

Since $B = \dfrac{\Phi}{A}$, cross-sectional area $A = \dfrac{\Phi}{B} = \dfrac{353 \times 10^{-3}}{1.8}$ m^2

$= 0.1961$ m^2

The pole face is circular, hence area $= \pi r^2$, where r is the radius.

Hence $\pi r^2 = 0.1961$

from which $r^2 = \dfrac{0.1961}{\pi}$ and radius $r = \sqrt{\left(\dfrac{0.1961}{\pi}\right)} = 0.250$ m

i.e. **the radius of the pole face is 250 mm**

6.4 Magnetomotive force and magnetic field strength

Magnetomotive force (mmf) is the cause of the existence of a magnetic flux in a magnetic circuit.

$$\textbf{mmf}, F_m = NI \textbf{ amperes}$$

where N is the number of conductors (or turns) and I is the current in amperes. The unit of mmf is sometimes expressed as 'ampere-turns'. However since 'turns' have no dimensions, the SI unit of mmf is the ampere.

Magnetic field strength (or **magnetizing force)** is given as

$$H = NI/l \textbf{ ampere per metre,}$$

where l is the mean length of the flux path in metres.

Thus **mmf** $= NI = Hl$ **amperes**.

Problem 3. A magnetizing force of 8000 A/m is applied to a circular magnetic circuit of mean diameter 30 cm by passing a current through a coil wound on the circuit. If the coil is uniformly wound around the circuit and has 750 turns, find the current in the coil.

$H = 8000$ A/m; $l = \pi d = \pi \times 30 \times 10^{-2}$ m; $N = 750$ turns

Since $H = \dfrac{NI}{l}$ then, $I = \dfrac{Hl}{N} = \dfrac{8000 \times \pi \times 30 \times 10^{-2}}{750}$

Thus, **current** $I = 10.05$ **A**

6.5 Permeability and B –H curves

For air, or any non-magnetic medium, the ratio of magnetic flux density to magnetizing force is a constant, i.e. $B/H =$ a constant. This constant is μ_0, the **permeability of free space** (or the magnetic space constant) and is equal to $4\pi \times 10^{-7}$ H/m, i.e., **for air, or any non-magnetic medium**, the ratio $\boxed{B/H = \mu_0}$ (Although all non-magnetic materials, including air, exhibit slight magnetic properties, these can effectively be neglected.)

For all media other than free space, $\boxed{B/H = \mu_0\mu_r}$

where u_r is the relative permeability, and is defined as

$$\boxed{\mu_r = \frac{\textbf{flux density in material}}{\textbf{flux density in a vacuum}}}$$

μ_r varies with the type of magnetic material and, since it is a ratio of flux densities, it has no unit. From its definition, μ_r for a vacuum is 1. $\mu_0\mu_r = \mu$, called the **absolute permeability**

By plotting measured values of flux density B against magnetic field strength H, a **magnetization curve** (or B–H **curve**) is produced. For nonmagnetic materials this is a straight line. Typical curves for four magnetic materials are shown in Figure 7.3.

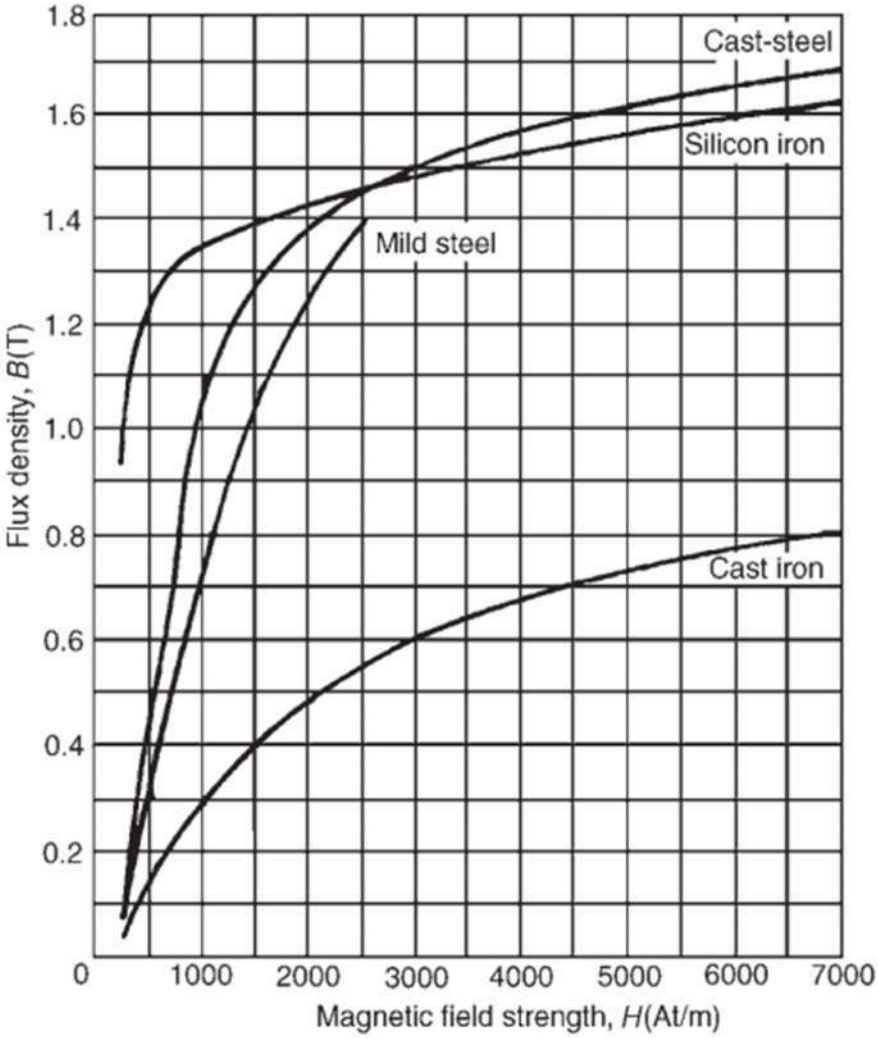


Figure 7.3 *B–H curves for four materials*

Cast iron	$\mu_r = 100–250$	Mild steel	$\mu_r = 200–800$
Silicon iron	$\mu_r = 1000–5000$	Cast steel	$\mu_r = 300–900$
Mumetal	$\mu_r = 200–5000$	Stalloy	$\mu_r = 500–6000$

Problem 5. Determine the magnetic field strength and the mmf required to produce a flux density of 0.25 T in an air gap of length 12 mm.

For air: $B = \mu_0 H$ (since $\mu_r = 1$)

$$\textbf{Magnetic field strength } H = \frac{B}{\mu_0} = \frac{0.25}{4\pi \times 10^{-7}} = \textbf{198 940 A/m}$$

$$\textbf{mmf} = Hl = 198\ 940 \times 12 \times 10^{-3} = \textbf{2387 A}$$

> Problem 7. An iron ring of mean diameter 10 cm is uniformly wound with 2000 turns of wire. When a current of 0.25 A is passed through the coil a flux density of 0.4 T is set up in the iron. Find (a) the magnetizing force and (b) the relative permeability of the iron under these conditions.

$l = \pi d = \pi \times 10$ cm $= \pi \times 10 \times 10^{-2}$ m; $N = 2000$ turns; $I = 0.25$ A; $B = 0.4$ T

(a) $$\mathbf{H} = \frac{NI}{l} = \frac{2000 \times 0.25}{\pi \times 10 \times 10^{-2}} = \frac{5000}{\pi} = \textbf{1592 A/m}$$

(b) $B = \mu_0 \mu_r H$, hence $$\boldsymbol{\mu_r} = \frac{B}{\mu_0 H} = \frac{0.4}{(4\pi \times 10^{-7})(1592)} = \textbf{200}$$

6.6 Reluctance

Reluctance *S* (or R_M) is the 'magnetic resistance' of a magnetic circuit to the presence of magnetic flux.

$$\textbf{Reluctance } S = \frac{F_M}{\Phi} = \frac{NI}{\Phi} = \frac{Hl}{BA} = \frac{l}{(B/H)A} = \frac{\boldsymbol{l}}{\boldsymbol{\mu_0 \mu_r A}}$$

The unit of reluctance is 1/H (or H^{-1}) or A/Wb

Stated alternatively,

$$\text{m.m.f.} = \text{flux} \times \text{reluctance}$$

$$F = \Phi S$$

or $$NI = \Phi S$$

It is clear that m.m.f. (F) is analogous to e.m.f. (E) and flux (Φ) is analogous to current (I) in a d.c. resistive circuit, where

$$\text{e.m.f.} = \text{current} \times \text{resistance}$$

$$E = IR$$

Consequently, *m.m.f = flux * reluctance* can be regarded as '*Ohm's law for a magnetic circuit*', since the m.m.f. (F), the total number of ampere-turns (NI) acting on the magnetic circuit.

Ferromagnetic materials have a low reluctance and can be used as **magnetic screens** to prevent magnetic fields affecting materials within the screen.

> Problem 10. A mild steel ring has a radius of 50 mm and a cross-sectional area of 400 mm^2. A current of 0.5 A flows in a coil wound uniformly around the ring and the flux produced is 0.1 mWb. If the relative permeability at this value of current is 200 find (a) the reluctance of the mild steel and (b) the number of turns on the coil.

$l = 2\pi r = 2 \times \pi \times 50 \times 10^{-3}$ m; $A = 400 \times 10^{-6}$ m^2; $I = 0.5$ A;
$\Phi = 0.1 \times 10^{-3}$ Wb; $\mu_r = 200$

(a) **Reluctance** $S = \dfrac{l}{\mu_0 \mu_r A} = \dfrac{2 \times \pi \times 50 \times 10^{-3}}{(4\pi \times 10^{-7})(200)(400 \times 10^{-6})}$

$$= \mathbf{3.125 \times 10^6/H}$$

(b) $S = \dfrac{\text{mmf}}{\Phi}$ i.e. $\text{mmf} = S\Phi$

so that $NI = S\Phi$ and

$$\text{hence } N = \frac{S\Phi}{I} = \frac{3.125 \times 10^6 \times 0.1 \times 10^{-3}}{0.5} = \mathbf{625 \text{ turns}}$$

6.7 Composite series magnetic circuits

For a series magnetic circuit having n parts, the **total reluctance *S*** is given by:

$$\boldsymbol{S = S_1 + S_2 + \ldots + S_n}$$

Problem 11. A closed magnetic circuit of cast steel contains a 6 cm long path of cross-sectional area 1 cm^2 and a 2 cm path of cross-sectional area 0.5 cm^2. A coil of 200 turns is wound around the 6 cm length of the circuit and a current of 0.4 A flows. Determine the flux density in the 2 cm path, if the relative permeability of the cast steel is 750.

For the 6 cm long path:

$$\text{Reluctance } S_1 = \frac{l_1}{\mu_0 \mu_r A_1} = \frac{6 \times 10^{-2}}{(4\pi \times 10^{-7})(750)(1 \times 10^{-4})}$$

$$= 6.366 \times 10^5/\text{H}$$

For the 2 cm long path:

$$\text{Reluctance } S_2 = \frac{l_2}{\mu_0 \mu_r A_2} = \frac{2 \times 10^{-2}}{(4\pi \times 10^{-7})(750)(0.5 \times 10^{-4})}$$

$$= 4.244 \times 10^5/\text{H}$$

$$S = \frac{\text{mmf}}{\Phi}, \text{ i.e. } \Phi = \frac{\text{mmf}}{S} = \frac{NI}{S} = \frac{200 \times 0.4}{10.61 \times 10^5} = 7.54 \times 10^{-5} \text{ Wb}$$

$$\text{Flux density in the 2 cm path, } B = \frac{\Phi}{A} = \frac{7.54 \times 10^{-5}}{0.5 \times 10^{-4}} = \mathbf{1.51\ T}$$

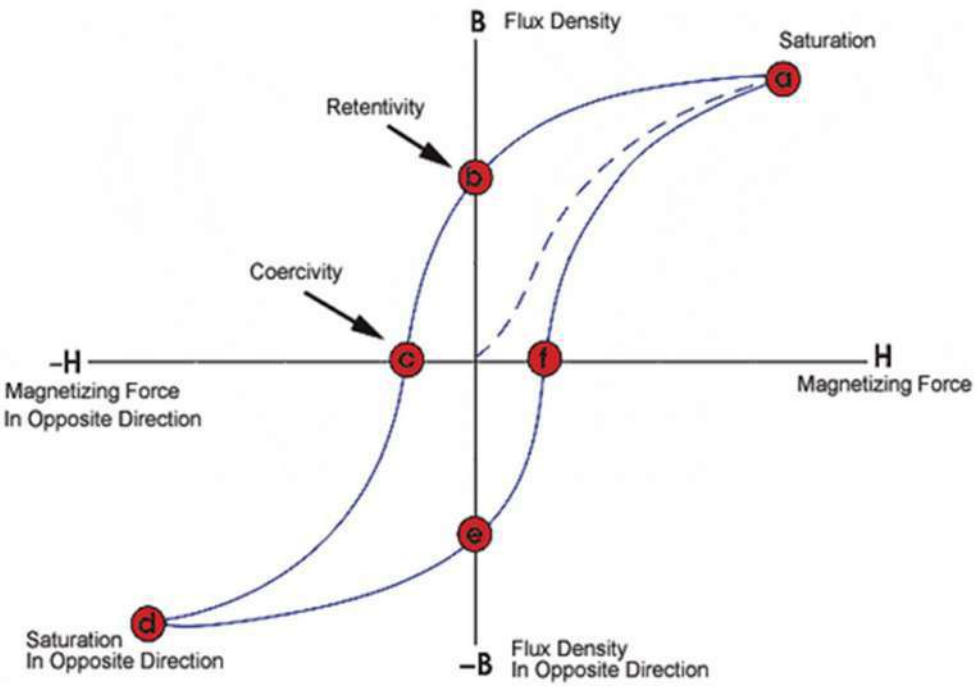
6.8 Comparison between electrical and magnetic quantities

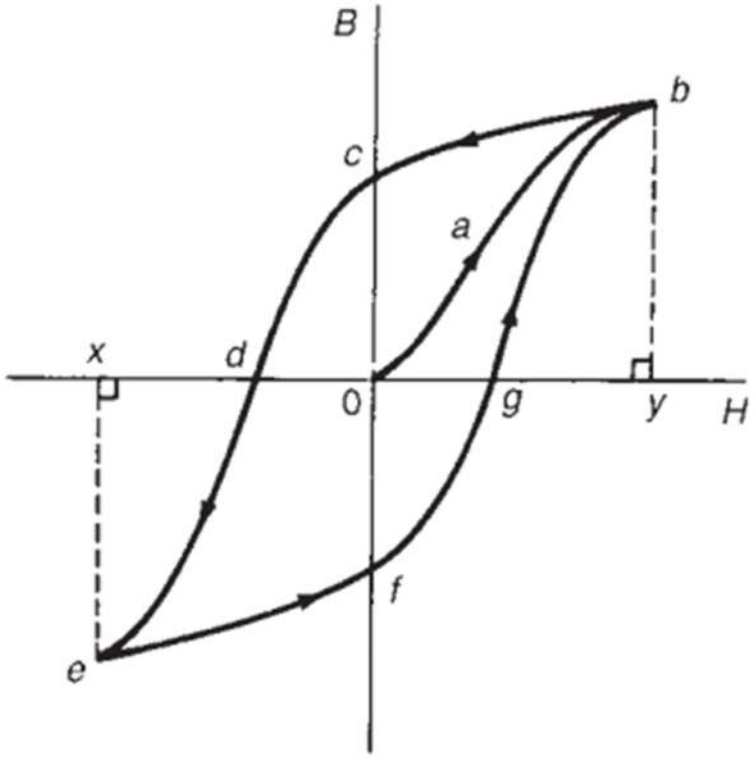
Electrical circuit	Magnetic circuit
e.m.f. E (V)	mmf F_m (A)
current I (A)	flux Φ (Wb)
resistance R (Ω)	reluctance S (H^{-1})
$I = \frac{E}{R}$	$\Phi = \frac{\text{mmf}}{S}$
$R = \frac{\rho l}{A}$	$S = \frac{l}{\mu_0 \mu_r A}$

Electrostatics		Electromagnetism	
Term	**Symbol**	**Term**	**Symbol**
Electric flux	Q	Magnetic flux	Φ
Electric flux density	D	Magnetix flux density	B
Electric field strength	E	Magnetic field strength	H
Electromotive force	E	Magnetomotive force	F
Electric potential difference	V	Magnetic potential difference	—
Permittivity of free space	ϵ_0	Permeability of free space	μ_0
Relative permittivity	ϵ_r	Relative permeability	μ_r
Absolute permittivity $= \dfrac{\text{electric flux density}}{\text{electric field strength}}$		Absolute permeability $= \dfrac{\text{magnetic flux density}}{\text{magnetic field strength}}$	
i.e. $\epsilon_0\epsilon_r = \epsilon = D/E$		i.e. $\mu_0\mu_r = \mu = B/H$	

6.9 Hysteresis and hysteresis loss

Hysteresis is the 'lagging' effect of flux density B whenever there are changes in the magnetic field strength H. When an **initially unmagnetised ferromagnetic material** is subjected to a varying magnetic field strength H, the flux density B produced in the material varies as shown below, the arrows indicating the direction of the cycle.





The figure (s) above is known as a **hysteresis loop**.

Let a ferromagnetic material which is completely demagnetized, i.e., one in which $B = H = 0$, be subjected to increasing values of magnetic field strength H and the corresponding flux density B measured. The domains begin to align and the resulting relationship between B and H is shown by the curve Oab in the figure above. At a particular value of H, shown as Oy, most of the domains will be aligned and it becomes difficult to increase the flux density any further. The material is said to be saturated. Thus by is the **saturation flux density.**

If the value of H is now reduced it is found that the flux density follows curve bc, i.e., the domains will tend to stay aligned even when the field is removed. When H is reduced to zero, flux remains in the iron. This **remanent flux density** or **remanence** is shown as Oc.

When H is increased in the opposite direction, the domains begin to realign in the opposite direction and the flux density decreases until, at a value shown as Od, the flux density has been reduced to zero. The magnetic field strength Od required to remove the residual magnetism, i.e., reduce B to zero, is called the **coercive force**.

Further increase of H in the reverse direction causes the flux density to increase in the reverse direction until saturation is reached, as shown by curve de. If the reversed magnetic field strength Ox is adjusted to the same value of Oy in the initial direction, then the final flux density xe is the same as yb. If H is varied backwards from Ox to Oy, the flux density follows the curve $efgb$, similar to curve $bcde$.

It is seen from the figure that the flux density changes lag behind the changes in the magnetic field strength. This effect is called **hysteresis.** The closed figure $bcdefgb$ is called the **hysteresis loop** (or the B/H loop).

A disturbance in the alignment of the domains of a ferromagnetic material causes energy to be expended in taking it through a cycle of magnetization. This energy appears as heat in the specimen and is called the **hysteresis loss**. **The energy loss associated with hysteresis is proportional to the area of the hysteresis loop**. The area of a hysteresis loop varies with the type of material. The area, and thus the energy loss, is much greater for hard materials than for soft materials.

For AC - excited devices the hysteresis loop is repeated every cycle of alternating current. Thus a hysteresis loop with a large area (as with hard steel) is often unsuitable since

the energy loss would be considerable. **Silicon steel has a narrow hysteresis loop, and thus small hysteresis loss, and is suitable for transformer cores and rotating machine armatures**.

6.10 Magnetic properties of materials

Bohr and Rutherford, who discovered atomic structure, suggested that electrons move around the nucleus confined to a plane, like planets around the sun.

Magnetic effects in materials are due to the electrons contained in them, the electrons giving rise to magnetism in the following two ways:

(i). by revolving around the nucleus
(ii). by their angular momentum about their own axis, called spin.

In each of these cases the charge of the electron can be thought of as moving round in a closed loop and therefore acting as a current loop. The main measurable quantity of an atomic model is the **magnetic moment.**

When applied to a loop of wire carrying a current,

$$\text{magnetic moment} = \text{current} \times \text{area of the loop}$$

Electrons associated with atoms possess magnetic moment which gives rise to their magnetic properties.

Diamagnetism is a phenomenon exhibited by materials having a **relative permeability less than unity** (**one**). When electrons move more or less in a spherical orbit around the nucleus, the magnetic moment due to this orbital is zero, all the current due to moving electrons being considered as averaging to zero. If the net magnetic moment of the electron spins were also zero then there would be no tendency for the electron motion to line up in the presence of a magnetic field.

Relative permeability is defined as

$$\frac{\text{flux density in material}}{\text{flux density in vacuum}}$$

Paramagnetism is a phenomenon exhibited by materials where the **relative permeability is greater than unity**. Paramagnetism occurs in substances where atoms have a permanent magnetic moment. This may be caused by the orbitals not being spherical or by the spin of the electrons. Electron spins tend to pair up and cancel each other. However, there are many atoms with odd numbers of electrons, or in which pairing is incomplete. Such atoms have what is called a permanent dipole moment. When a field is applied to them they tend to line up with the field, like compass needles, and so strengthen the flux in that region. (Diamagnetic materials do not tend to line up with the field in this way.) When this effect is stronger than the diamagnetic effect, the overall effect is to make the relative permeability greater than one. Such materials are called paramagnetic. Examples are **manganese, chromium or aluminium**.

Ferromagnetism is the phenomenon exhibited by materials having a relative permeability which is considerably greater than 1 and which varies with flux density. **Iron, cobalt and nickel** are the only elements that are ferromagnetic at ordinary working temperatures, but there are several alloys containing one or more of these metals as constituents, with widely varying ferromagnetic properties.

Consider the simple model of a single iron atom represented in Figure 38.1. It consists of a small heavy central nucleus surrounded by a total of 26 electrons. Each electron has an orbital motion about the nucleus in a limited region, or shell, such shells being represented by circles ***K***, ***L***, ***M*** and ***N***. The numbers in Figure 38.1 represent the number of electrons in each shell.

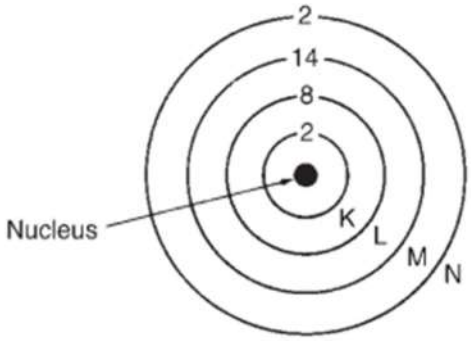


Figure 38.1 *Single iron atom*

The outer shell ***N*** contains two loosely held electrons, these electrons becoming the carriers of electric current, making iron electrically conductive. **There are 14 electrons in the *M* shell and it is this group that is responsible for magnetism**. An electron carries a negative charge and a charge in motion constitutes an electric current with which is associated a magnetic field. Magnetism would therefore result from the orbital motion of each electron in the atom. **However, experimental evidence indicates that the resultant magnetic effect due to all the orbital motions in the metal solid is zero; thus the orbital currents may be disregarded**.

In addition to the orbital motion, each electron spins on its own axis. A rotating charge is equivalent to a circular current and gives rise to a magnetic field. In any atom, all the axes about which the electrons spin are parallel, but rotation may be in either direction. **In the single atom shown in Figure 38.1, in each of the K, L and N shells equal numbers of electrons spin in the clockwise and anticlockwise directions respectively and therefore these shells are magnetically neutral. However, in shell M, nine of the electrons spin in one direction while five spin in the opposite direction. There is therefore a resultant effect due to four electrons.**

The atom of cobalt has 15 electrons in the M shell, nine spinning in one direction and six in the other. Thus with cobalt there is a resultant effect due to 3 electrons. A nickel atom has a resultant effect due to 2 electrons. The atoms of the paramagnetic elements, such as manganese, chromium or aluminium, also have a resultant effect for the same reasons as that of iron, cobalt and nickel. However, in the **diamagnetic materials there is an exact equality between the clockwise and anticlockwise spins**.

The total magnetic field of the resultant effect due to the four electrons in the iron atom is large enough to influence other atoms. Thus the orientation of one atom tends to spread through the material, with atoms acting together in groups instead of behaving independently. These groups of atoms, called **domains** (which tend to remain permanently magnetized), act as units. Thus, when a field is applied to a piece of iron, these domains as a whole tend to line up and large flux densities can be produced. This means that the relative permeability of such materials is much greater than one. **As the applied field is increased, more and more domains align and the induced flux increases.**

Ferromagnetic effects decrease with temperature, as do those due to paramagnetism. The loss of ferromagnetism with temperature is more sudden, however; the temperature at which it has all disappeared is called the **Curie temperature**. The ferromagnetic properties reappear on cooling, but any magnetism will have disappeared. Thus a permanent magnet will be demagnetized by heating above the Curie temperature (1040K for iron) but can be remagnetized after cooling. Above the Curie temperature, ferromagnetics behave as paramagnetics.

Nonpermanent magnetic materials are those in which magnetism may be induced. Examples: **Silicon-iron alloys, Nickel-iron alloys.**

A permanent magnet is one in which the material used exhibits magnetism without the need for excitation by a current-carrying coil. The **silicon-iron and nickel-iron alloys** are **'soft'** magnetic materials having high permeability and hence low hysteresis loss. The opposite characteristics are required in the '**hard'** materials used to make permanent magnets. In permanent magnets, high remanent flux density and high coercive force, after magnetization to saturation, are desirable in order to resist demagnetization.

6.11 Revision Questions

(Where appropriate, assume $\mu_0 = 4\pi \times 10^{-7}$ H/m)

1 What is the flux density in a magnetic field of cross-sectional area 20 cm^2 having a flux of 3 mWb? [1.5 T]

2 Determine the total flux emerging from a magnetic pole face having dimensions 5 cm by 6 cm, if the flux density is 0.9 T. [2.7 mWb]

3 The maximum working flux density of a lifting electromagnet is 1.9 T and the effective area of a pole face is circular in cross-section. If the total magnetic flux produced is 611 mWb determine the radius of the pole face. [32 cm]

4 Find the magnetic field strength and the magnetomotive force needed to produce a flux density of 0.33 T in an air-gap of length 15 mm. [(a) 262 600 A/m (b) 3939 A]

5 An air-gap between two pole pieces is 20 mm in length and the area of the flux path across the gap is 5 cm^2. If the flux required in the air-gap is 0.75 mWb find the mmf necessary. [23 870 A]

6 Find the magnetic field strength applied to a magnetic circuit of mean length 50 cm when a coil of 400 turns is applied to it carrying a current of 1.2 A. [960 A/m]

7 A solenoid 20 cm long is wound with 500 turns of wire. Find the current required to establish a magnetizing force of 2500 A/m inside the solenoid. [1 A]

8 A magnetic field strength of 5000 A/m is applied to a circular magnetic circuit of mean diameter 250 mm. If the coil has 500 turns find the current in the coil. [7.85 A]

9 Find the relative permeability of a piece of silicon iron if a flux density of 1.3 T is produced by a magnetic field strength of 700 A/m [1478]

10 Part of a magnetic circuit is made from steel of length 120 mm, cross-sectional area 15 cm^2 and relative permeability 800. Calculate (a) the reluctance and (b) the absolute permeability of the steel. [(a) 79 580 /H (b) 1 mH/m]

11 A steel ring of mean diameter 120 mm is uniformly wound with 1500 turns of wire. When a current of 0.30 A is passed through the coil a flux density of 1.5 T is set up in the steel. Find the relative permeability of the steel under these conditions. [1000]

12 A mild steel closed magnetic circuit has a mean length of 75 mm and a cross-sectional area of 320.2 mm^2. A current of 0.40 A flows in a coil wound uniformly around the circuit and the flux produced is 200 μWb. If the relative permeability of the steel at this value of current is 400 find (a) the reluctance of the material and (b) the number of turns of the coil. [(a) 466 000/H (b) 233]

7 Electromagnetism

An electric current flowing through a conductor induces a magnetic field around the conductor. The direction of the magnetic lines of flux is best remembered by the **screw rule**. This states that:

'*If a normal right-hand thread screw (woodscrew) is screwed along the conductor in the direction of the current, the direction of rotation of the screw is in the direction of the magnetic field.*'

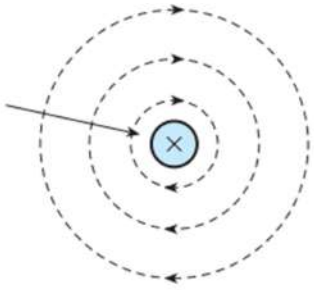


Fig. 6.7 Magnetic flux due to current in a straight conductor

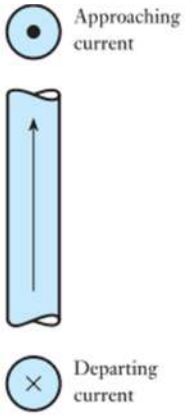


Fig. 6.8 Current conventions

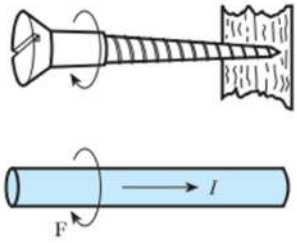
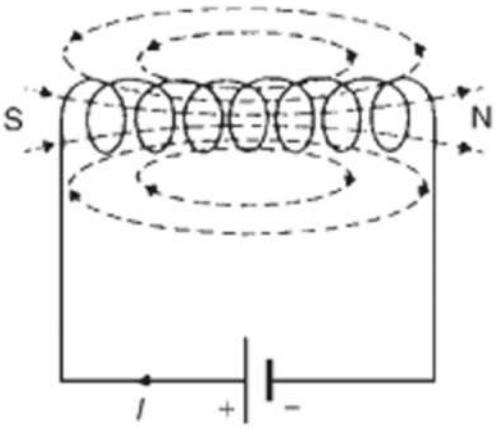


Fig. 6.9 Right-hand screw rule

An alternative method of deriving this relationship is to ***grip*** *the conductor with the* ***right hand****, with the* ***thumb outstretched parallel to the conductor and pointing in the direction of the current; the fingers then point in the direction of the magnetic flux around the conductor****.*

If the conductor is a solenoid, the direction of the magnetic field produced by a current in the solenoid may be deduced by applying either the **screw** or the **grip rule**:

If the axis of the **screw** is placed along that of the solenoid and if the screw is turned in the direction of the current, it travels in the direction of the magnetic field ***inside*** the solenoid, namely towards the right in Fig. 6.10.



(a) Magnetic field of a solenoid

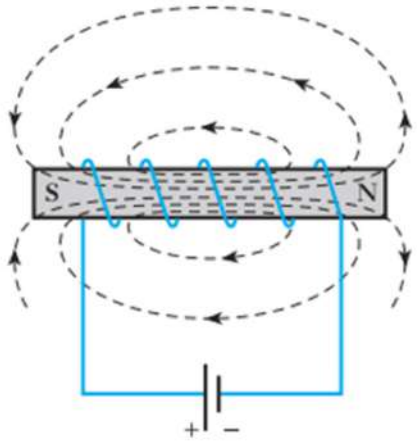


Fig. 6.10 Solenoid with a steel core

The grip rule states that if the coil (solenoid) is gripped with the **right** hand, with the fingers pointing in the direction of the current, then the thumb, outstretched parallel to the axis of the solenoid, points in the direction of the magnetic field ***inside*** the solenoid.

Magnets (such as the solenoid of Fig. 6.10 above) that result from currents are called **electromagnets**.

7.1 Force on a current-carrying conductor

If a current-carrying conductor is placed in a magnetic field produced by permanent magnets, then the fields due to the current-carrying conductor and the permanent magnets **interact** and cause a force to be exerted on the conductor. The force on the current-carrying conductor in a magnetic field depends upon:

a). the flux density of the field, ***B*** teslas.
b). the strength of the current, ***I*** amperes.
c). the length of the conductor perpendicular to the magnetic field, ***l*** metres.
d). the directions of the field and the current.

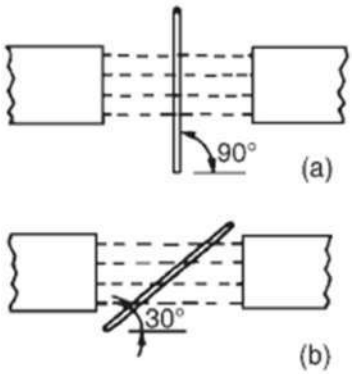
Thus,

When the magnetic field, the current and the conductor are mutually at right angles then:

Force $F = BIl$ newtons

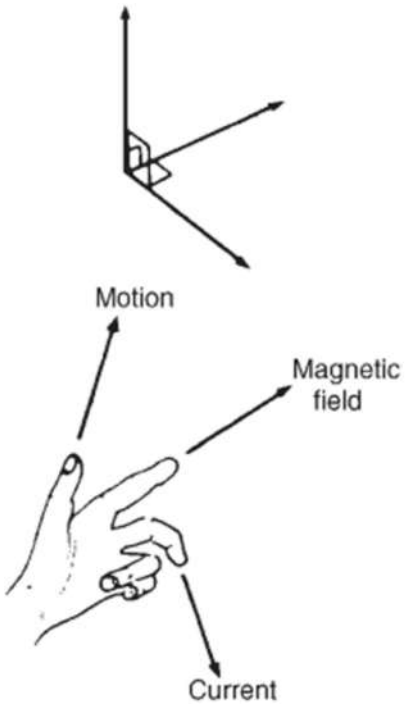
When the conductor and the field are at an angle $\theta°$ to each other then:

Force $F = BIl \sin\theta$ newtons



The **direction of the force** exerted on a conductor can be predetermined by using **Fleming's left-hand rule** (often called the **motor rule**) which states:

Let the thumb, first finger and second finger of the left hand be extended such that they are all at right-angles to each other, (as shown below). If the first finger points in the direction of the magnetic field, the second finger points in the direction of the current, then the thumb will point in the direction of the motion of the conductor.



Summarizing:
First finger - Field
SeCond finger - Current
ThuMb - Motion

Problem 4. A conductor 350 mm long carries a current of 10 A and is at right-angles to a magnetic field lying between two circular pole faces each of radius 60 mm. If the total flux between the pole faces is 0.5 mWb, calculate the magnitude of the force exerted on the conductor.

$l = 350$ mm $= 0.35$ m; $I = 10$ A;

Area of pole face $A = \pi r^2 = \pi(0.06)^2$ m^2;

$\Phi = 0.5$ mWb $= 0.5 \times 10^{-3}$ Wb

Force $F = BIl$, and $B = \dfrac{\Phi}{A}$

hence force $F = \left(\dfrac{\Phi}{A}\right) Il = \dfrac{(0.5 \times 10^{-3})}{\pi(0.06)^2}(10)(0.35)$ newtons

i.e. force = 0.155 N

Problem 6. A coil is wound on a rectangular former of width 24 mm and length 30 mm. The former is pivoted about an axis passing through the middle of the two shorter sides and is placed in a uniform magnetic field of flux density 0.8 T, the axis being perpendicular to the field. If the coil carries a current of 50 mA, determine the force on each coil side (a) for a single-turn coil, (b) for a coil wound with 300 turns.

(a) Flux density $B = 0.8$ T; length of conductor lying at right-angles to field $l = 30$ mm $= 30 \times 10^{-3}$ m; current $I = 50$ mA $= 50 \times 10^{-3}$ A

For a single-turn coil, force on each coil side

$$F = BIl = 0.8 \times 50 \times 10^{-3} \times 30 \times 10^{-3}$$

$$= \mathbf{1.2 \times 10^{-3}\ N},\ \text{or}\ \mathbf{0.0012\ N}$$

(b) When there are 300 turns on the coil there are effectively 300 parallel conductors each carrying a current of 50 mA. Thus the total force produced by the current is 300 times that for a single-turn coil. Hence force on coil side $F = 300\ BIl = 300 \times 0.0012 = \mathbf{0.36\ N}$

7.2 Force on a charge

When a charge of $\boldsymbol{Q}$ coulombs is moving at a velocity of $\boldsymbol{v}$ **m/s** in a magnetic field of flux density $\boldsymbol{B}$ teslas, the charge moving perpendicular to the field, then the magnitude of the force $\boldsymbol{F}$ exerted on the charge is given by:

$$\boldsymbol{F = QvB}\ \textbf{newtons}$$

Problem 17. An electron in a television tube has a charge of 1.6×10^{-19} coulombs and travels at 3×10^{7} m/s perpendicular to a field of flux density 18.5 μT. Determine the force exerted on the electron in the field.

From above, force $F = QvB$ newtons, where
Q = charge in coulombs = 1.6×10^{-19} C;
v = velocity of charge = 3×10^{7} m/s;
and B = flux density = 18.5×10^{-6} T

Hence force on electron $F = 1.6 \times 10^{-19} \times 3 \times 10^{7} \times 18.5 \times 10^{-6}$
$= 1.6 \times 3 \times 18.5 \times 10^{-18}$
$= 88.8 \times 10^{-18} = \mathbf{8.88 \times 10^{-17}}$ **N**

7.3 Electromagnetic induction

When a conductor is moved across a magnetic field so as to cut through the lines of force (or flux), an electromotive force (e.m.f.) is produced in the conductor. **If the conductor forms part of a closed circuit then the e.m.f. produced causes an electric current to flow round the circuit**. Hence an e.m.f. (and thus current) is 'induced' in the conductor as a result of its movement across the magnetic field. **It is the relative movement of the magnetic flux and the conductor that causes an e.m.f. and thus current, to be induced in the coil**. This effect is known as '**electromagnetic induction**'.

7.3.1 Laws of electromagnetic induction

Faraday's laws of electromagnetic induction state:

(i). 'An induced e.m.f. is set up whenever the magnetic field linking that circuit changes.'

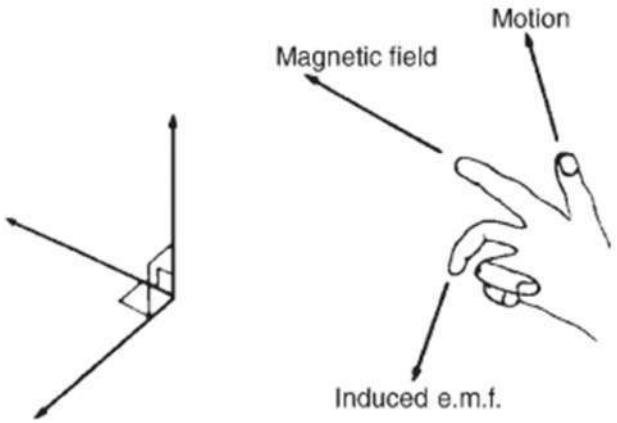
(ii). 'The magnitude of the induced e.m.f. in any circuit is proportional to the rate of change of the magnetic flux linking the circuit.'

Lenz's law states:

'*The direction of an induced e.m.f. is always such that it tends to set up a current opposing the motion or the change of flux responsible for inducing that e.m.f.*'

An alternative method to Lenz's law of determining relative directions is given by **Fleming's Right-hand rule** (often called the **geneRator rule**) which states:

Let the thumb, first finger and second finger of the right hand be extended such that they are all at right angles to each other (as shown below). If the first finger points in the direction of the magnetic field, the thumb points in the direction of motion of the conductor relative to the magnetic field, then the second finger will point in the direction of the induced e.m.f.



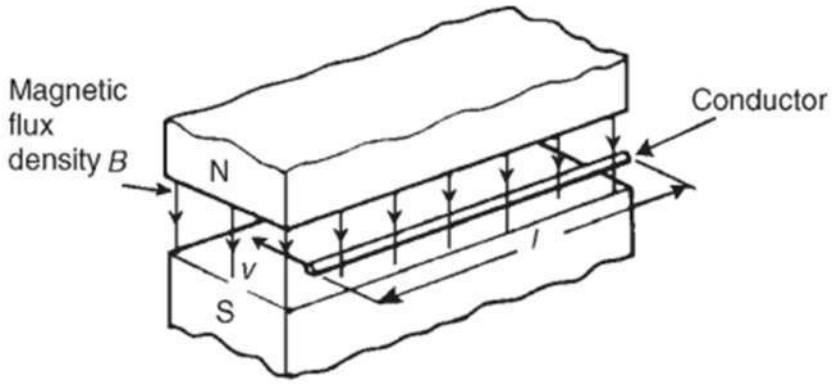
Summarizing:

First finger — Field

ThuMb — Motion

SEcond finger — E.m.f.

In a generator, conductors forming an electric circuit are made to move through a magnetic field. By Faraday's law an e.m.f. is induced in the conductors and thus a source of e.m.f. is created.



A generator converts mechanical energy into electrical energy. The induced e.m.f. E set up between the ends of the conductor shown above is given by:

$$E = Blv \text{ volts,}$$

where $\boldsymbol{B}$, the flux density, is measured in teslas, $\boldsymbol{l}$, the length of conductor in the magnetic field, is measured in metres, and $\boldsymbol{v}$, the conductor velocity, is measured in metres per second.

If the conductor moves at an angle $\boldsymbol{\theta}°$ to the magnetic field (instead of at 90° as assumed above) then

$$\boldsymbol{E = Blv \sin\theta \text{ volts}}$$

Problem 1. A conductor 300 mm long moves at a uniform speed of 4 m/s at right-angles to a uniform magnetic field of flux density 1.25 T. Determine the current flowing in the conductor when (a) its ends are open-circuited, (b) its ends are connected to a load of 20 Ω resistance.

When a conductor moves in a magnetic field it will have an e.m.f. induced in it but this e.m.f. can only produce a current if there is a closed circuit.

Induced e.m.f. $E = Blv = (1.25)\left(\frac{300}{1000}\right)(4) = 1.5$ V

(a) If the ends of the conductor are open circuited no current will flow even though 1.5 V has been induced.

(b) From Ohm's law, $I = \frac{E}{R} = \frac{1.5}{20} = \mathbf{0.075\ A}$ or $\mathbf{75\ mA}$

Problem 4. The wing span of a metal aeroplane is 36 m. If the aeroplane is flying at 400 km/h, determine the e.m.f. induced between its wing tips. Assume the vertical component of the earth's magnetic field is 40 μT

Induced e.m.f. across wing tips, $E = Blv$

$B = 40\ \mu\text{T} = 40 \times 10^{-6}$ T; $l = 36$ m

$$v = 400\frac{\text{km}}{\text{h}} \times 1000\frac{\text{m}}{\text{km}} \times \frac{1\ \text{h}}{60 \times 60\ \text{s}} = \frac{(400)(1000)}{3600}$$

$$= \frac{4000}{36}\ \text{m/s}$$

$$\text{Hence } E = Blv = (40 \times 10^{-6})(36)\left(\frac{4000}{36}\right)$$

$$= \mathbf{0.16\ V}$$

7.3.2 Inductance

When the e.m.f. is induced in the same circuit as that in which the current is changing, the property is called **self-inductance**, ***L*** When the e.m.f. is induced in a circuit by a change of flux due to current changing in an adjacent circuit, the property is called **mutual inductance**, ***M***.

Induced e.m.f. in a coil of N turns,

$$E = -N\frac{d\Phi}{dt} \text{ volts,}$$

where $d\Phi$ is the change in flux in Webers, and dt is the time taken for the flux to change in seconds(i.e., $d\Phi/dt$ is the rate of change of flux).

Induced e.m.f. in a coil of inductance L henrys,

$$E = -L\frac{dI}{dt} \text{ volts,}$$

where dI is the change in current in amperes and dt is the time taken for the current to change in seconds(i.e., dI/dt is the rate of change of current). The minus sign in each of the above two equations remind us of its direction (given by Lenz's law).

Problem 7. A flux of 400 μWb passing through a 150-turn coil is reversed in 40 ms. Find the average e.m.f. induced.

Since the flux reverses, the flux changes from +400 μWb to −400 μWb, a total change of flux of 800 μWb

$$\text{Induced e.m.f. } E = -N\frac{d\Phi}{dt} = -(150)\left(\frac{800 \times 10^{-6}}{40 \times 10^{-3}}\right)$$

$$= -\left(\frac{150 \times 800 \times 10^{3}}{40 \times 10^{6}}\right)$$

Hence **the average e.m.f. induced $E = -3$ volts**

Problem 9. An e.m.f. of 1.5 kV is induced in a coil when a current of 4 A collapses uniformly to zero in 8 ms. Determine the inductance of the coil.

Change in current, $dI = (4 - 0) = 4$ A; $dt = 8$ ms $= 8 \times 10^{-3}$ s;

$$\frac{dI}{dt} = \frac{4}{8 \times 10^{-3}} = \frac{4000}{8} = 500 \text{ A/s}; \quad E = 1.5 \text{ kV} = 1500 \text{ V}$$

Since $|E| = L\left(\frac{dI}{dt}\right)$, inductance, $L = \frac{|E|}{(dI/dt)} = \frac{1500}{500} = \mathbf{3\ H}$

(Note that $|E|$ means the 'magnitude of E', which disregards the minus sign)

7.3.2.1 Inductance of a coil

As seen above,

$$\boxed{E = -N\frac{d\Phi}{dt} \text{ volts,}} \quad \textbf{OR} \quad \boxed{E = -L\frac{dI}{dt} \text{ volts,}}$$

If a current changing from **0** to I amperes, produces a flux change from **0** to Φ Webers, then $dI = I$ and $d\Phi = \Phi$. Then, induced e.m.f. $E = N\Phi/t = LI/t$, from which

$$\textbf{inductance of coil, } \boxed{L = \frac{N\Phi}{I} \text{ henrys}}$$

Problem 13. A 750 turn coil of inductance 3 H carries a current of 2 A. Calculate the flux linking the coil and the e.m.f. induced in the coil when the current collapses to zero in 20 ms

Coil inductance, $L = \frac{N\Phi}{I}$ from which, flux $\Phi = \frac{LI}{N}$

$$= \frac{(3)(2)}{750} = 8 \times 10^{-3} = \mathbf{8\ mWb}$$

Induced e.m.f. $E = -L\left(\frac{dI}{dt}\right) = -3\left(\frac{2 - 0}{20 \times 10^{-3}}\right) = \mathbf{-300\ V}$

(Alternatively, $E = -N\frac{d\Phi}{dt} = -(750)\left(\frac{8 \times 10^{-3}}{20 \times 10^{-3}}\right) = \mathbf{-300\ V}$)

7.3.2.2 Mutual inductance

Mutually induced e.m.f. in the second coil,

$$E_2 = -M\frac{dI_1}{dt} \text{ volts,}$$

where M is the **mutual inductance** between two coils, in henrys, and dI_1/dt is the rate of change of current in the first coil.

Problem 14. Calculate the mutual inductance between two coils when a current changing at 200 A/s in one coil induces an e.m.f. of 1.5 V in the other.

Induced e.m.f. $|E_2| = M\frac{dI_1}{dt}$, i.e., $1.5 = M(200)$

Thus **mutual inductance**, $M = \frac{1.5}{200} = \mathbf{0.0075\ H\ or\ 7.5\ mH}$

Problem 16. Two coils have a mutual inductance of 0.2 H. If the current in one coil is changed from 10 A to 4 A in 10 ms, calculate (a) the average induced e.m.f. in the second coil, (b) the change of flux linked with the second coil if it is wound with 500 turns.

(a) Induced e.m.f. $E_2 = -M\frac{dI_1}{dt} = -(0.2)\left(\frac{10-4}{10\times10^{-3}}\right) = \mathbf{-120\ V}$

(b) Induced e.m.f. $|E_2| = N\frac{d\Phi}{dt}$, hence $d\Phi = \frac{|E_2|dt}{N}$

Thus the change of flux, $d\Phi = \frac{120(10\times10^{-3})}{500} = \mathbf{2.4\ mWb}$

7.4 Revision Questions

Induced e.m.f.

1 A conductor of length 15 cm is moved at 750 mm/s at right-angles to a uniform flux density of 1.2 T. Determine the e.m.f. induced in the conductor. [0.135 V]

2 Find the speed that a conductor of length 120 mm must be moved at right angles to a magnetic field of flux density 0.6 T to induce in it an e.m.f. of 1.8 V. [25 m/s]

3 A 25 cm long conductor moves at a uniform speed of 8 m/s through a uniform magnetic field of flux density 1.2 T. Determine the current flowing in the conductor when (a) its ends are open-circuited, (b) its ends are connected to a load of 15 ohms resistance. [(a) 0 (b) 0.16 A]

4 A car is travelling at 80 km/h. Assuming the back axle of the car is 1.76 m in length and the vertical component of the earth's magnetic field is 40 μT, find the e.m.f. generated in the axle due to motion. [1.56 mV]

Inductance

6 Find the e.m.f. induced in a coil of 200 turns when there is a change of flux of 30 mWb linking with it in 40 ms. [−150 V]

7 An e.m.f. of 25 V is induced in a coil of 300 turns when the flux linking with it changes by 12 mWb. Find the time, in milliseconds, in which the flux makes the change. [144 ms]

8 An ignition coil having 10 000 turns has an e.m.f. of 8 kV induced in it. What rate of change of flux is required for this to happen? [0.8 Wb/s]

9 A flux of 0.35 mWb passing through a 125-turn coil is reversed in 25 ms. Find the magnitude of the average e.m.f. induced. [3.5 V]

Inductance of a coil

12 A flux of 30 mWb links with a 1200 turn coil when a current of 5 A is passing through the coil. Calculate (a) the inductance of the coil, (b) the energy stored in the magnetic field, and (c) the average e.m.f. induced if the current is reduced to zero in 0.20 s. [(a) 7.2 H (b) 90 J (c) 180 V]

13 An e.m.f. of 2 kV is induced in a coil when a current of 5 A collapses uniformly to zero in 10 ms. Determine the inductance of the coil. [4 H]

14 An average e.m.f. of 60 V is induced in a coil of inductance 160 mH when a current of 7.5 A is reversed. Calculate the time taken for the current to reverse. [40 ms]

15 A coil of 2500 turns has a flux of 10 mWb linking with it when carrying a current of 2 A. Calculate the coil inductance and the e.m.f. induced in the coil when the current collapses to zero in 20 ms. [12.5 H, 1.25 kV]

16 A coil is wound with 600 turns and has a self inductance of 2.5 H. What current must flow to set up a flux of 20 mWb?

[4.8 A]

17 When a current of 2 A flows in a coil, the flux linking with the coil is 80 μWb. If the coil inductance is 0.5 H, calculate the number of turns of the coil. [12 500]

18 A steady current of 5 A when flowing in a coil of 1000 turns produces a magnetic flux of 500 μWb. Calculate the inductance of the coil. The current of 5 A is then reversed in 12.5 ms. Calculate the e.m.f. induced in the coil. [0.1 H, 80 V]

Mutual inductance

19 The mutual inductance between two coils is 150 mH. Find the magnitude of the e.m.f. induced in one coil when the current in the other is increasing at a rate of 30 A/s. [4.5 V]

20 Determine the mutual inductance between two coils when a current changing at 50 A/s in one coil induces an e.m.f. of 80 mV in the other. [1.6 mH]

21 Two coils have a mutual inductance of 0.75 H. Calculate the magnitude of the e.m.f. induced in one coil when a current of 2.5 A in the other coil is reversed in 15 ms. [250 V]

22 The mutual inductance between two coils is 240 mH. If the current in one coil changes from 15 A to 6 A in 12 ms, calculate (a) the average e.m.f. induced in the other coil, (b) the change of flux linked with the other coil if it is wound with 400 turns.

[(a) −180 V (b) 5.4 mWb]

8 Alternating voltages and currents

In general, we can classify electrical systems into three on the basis of variations of their currents / voltages:

1. Direct current systems - those in which current remains constant throughout.
2. Transient systems - those in which the current / voltage has varied for a short period of time.
3. Alternating systems – those in which the magnitudes of the voltage and of the current vary in a **repetitive manner**. Examples of such repetitive currents are shown below:

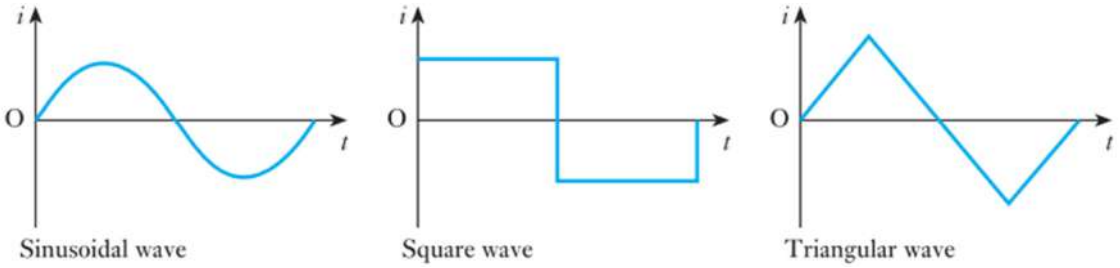


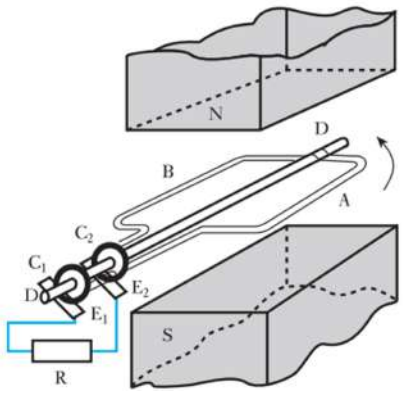
Fig. Alternating current waveforms

An alternating current flows first in one direction and then in the other. The cycle of variation is repeated exactly for each direction.

8.1 Generation of an alternating e.m.f.

Figure 9.2 shows a loop AB carried by a spindle DD rotated at a constant speed in an **anticlockwise** direction in a uniform magnetic field due to poles NS. The ends of the loop are brought out to two slip-rings C_1 and C_2, attached to but insulated from DD. Bearings on these rings are carbon brushes E_1 and E_2, which are connected to an external resistor R.

Fig. 9.2 Generation of an alternating e.m.f.



When the plane of the loop is horizontal, as shown in Fig. 9.3(a), the two sides A and B are moving parallel to the direction of the magnetic flux; it follows that no flux is being cut and no e.m.f. is being generated in the loop. Subsequent diagrams in Fig. 9.3 show the effects which occur as the coil is rotated. In Fig. 9.3(b), the coil sides are cutting the flux and therefore an e.m.f. is induced in the coil sides. **Since the coil sides are moving in opposite directions, the e.m.f.s act in opposite directions**, as shown by the dot and cross notation. **However, they do act in the same direction around the coil so that the e.m.f. which appears at the brushes is twice that which is induced in a coil side.** Once the coil reaches the position shown in Fig. 9.3(c), the rate of cutting reaches a maximum. Thereafter the e.m.f. falls to zero by the time the coil has rotated to the position shown in Fig. 9.3(d).

The induced e.m.f. in the position shown in Fig. 9.3(e) is of particular interest. At first sight, it appears that the diagram is the same as that of Fig. 9.3(b), but in fact it is side A which bears the cross while side B has the dot. This means that the e.m.f. is of the same magnitude but of the opposite polarity. This observation also applies to Fig. 9.3(f). It follows that the variation of induced e.m.f. during the second half of the cycle of rotation is the same in magnitude as during the first half but the polarity of the e.m.f. has reversed.

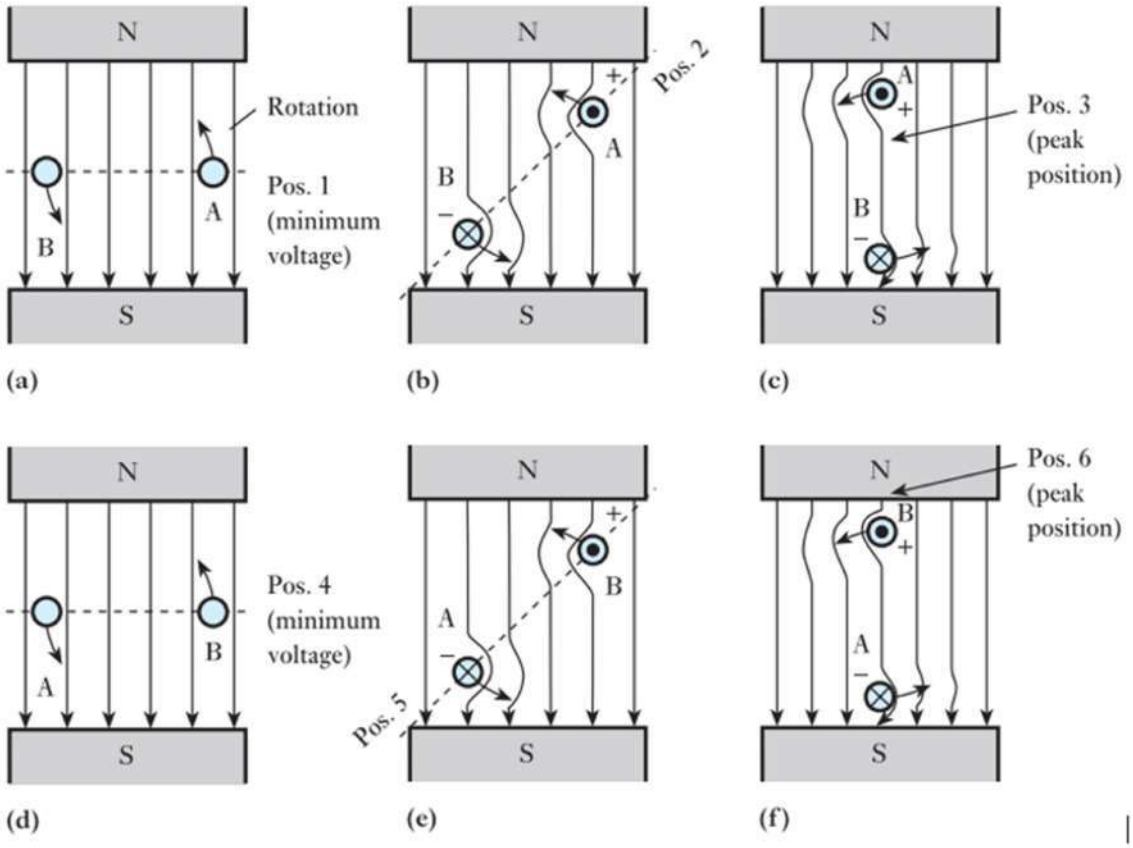


Fig. 9.3 EMF in rotating coil

As discussed previously, if ***B*** is the flux density in teslas and if ***l*** is the length in metres of each of the parallel sides A and B of the loop, it follows that the **e.m.f. generated in one side of loop** is

$$Blu \sin \theta \text{ volts}$$

and total e.m.f. generated in loop is

$$2Blu \sin \theta \text{ volts}$$

$$\therefore \quad e = 2Blu \sin \theta$$

That is, the generated e.m.f. is proportional to sin θ. When $\theta = 90°$, the plane of the loop is vertical and both sides of the loop are cutting the magnetic flux at the maximum rate, so that the generated e.m.f. is then at its maximum value, E_m. Hence, it follows that when $\theta = 90°$, $E_m = 2Blu$ volts.

If ***b*** is the breadth of the loop in metres, and ***n*** the speed of rotation in revolutions per second, then u is $\boldsymbol{\pi bn}$ metres per second and

$$E_m = 2Bl \times \pi bn \text{ volts}$$
$$= 2\pi BAn \text{ volts}$$

where

$$A = lb = \text{area of loop in square metres}$$

If the loop is replaced by a coil of N turns in series, each turn having an area of $\boldsymbol{A}$ square metres, maximum value of e.m.f. generated in coil is

$$E_m = 2\pi BAnN \text{ volts}$$

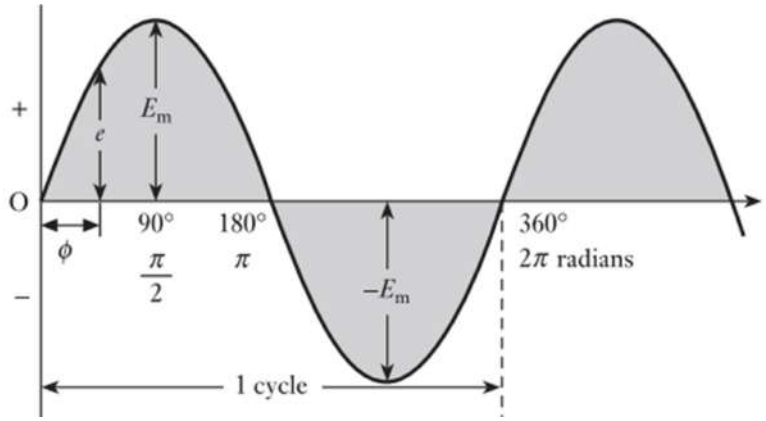
and instantaneous value of e.m.f. generated in coil is

$$e = E_m \sin\theta = 2\pi BAnN \sin\theta \text{ volts}$$

$$\therefore \quad e = 2\pi BAnN \sin\theta$$

This e.m.f. can be represented by a sine wave as below, where $\boldsymbol{E_m}$ represents the maximum value of the e.m.f. and e is the value after the loop has rotated through an angle $\boldsymbol{\theta}$ from the position of zero e.m.f.

Sine wave of e.m.f.



Lower-case letters are used to represent instantaneous values and uppercase letters represent definite values such as maximum, average or r.m.s. values. In a.c. circuits, capital *I* and *V* without any subscript represent r.m.s. values.

In the figure above, E_m represents the maximum value of the e.m.f. and e is the value after the loop has rotated through an angle θ from the position of zero e.m.f. When the loop has rotated through 180° or π radians, the e.m.f. is again zero. When θ is varying between 180° and 360° (π and 2π radians), side A of the loop is moving towards the right in Fig. 9.2 and is therefore cutting the magnetic flux in the opposite direction to that during the first half-revolution. Hence, if we regard the e.m.f. as positive while θ is varying between 0 and 180°, it is negative while θ is varying between 180° and 360°, i.e. when θ varies between 180° and 270°, the value of the e.m.f. increases from zero to $-E_m$ and then decreases to zero as θ varies between 270° and 360°. **Subsequent revolutions of the loop merely produce a repetition of the e.m.f. wave**.

8.2 Waveform terms and definitions

Waveform. The variation of a quantity such as voltage or current shown on a graph to a base of time or rotation is a waveform.

Cycle. Each repetition of a variable quantity, recurring at equal intervals, is termed a cycle.

Period. The duration of one cycle is termed its period.

Frequency. The number of cycles that occur in 1 second is termed the frequency of that quantity.

Instantaneous value. The magnitude of a waveform at any instant in time (or position of rotation). Instantaneous values are denoted by **lower-case symbols such as *e*, *v* and *i*.**

Peak value. The maximum instantaneous value measured from its zero value is known as its peak value.

Peak-to-peak value. The maximum variation between the maximum positive instantaneous value and the maximum negative instantaneous value is the peak-to-peak value. For a sinusoidal waveform, this is twice the peak value. The peak-to-peak value is $\boldsymbol{E_{pp}}$ **or** $\boldsymbol{V_{pp}}$ **or** $\boldsymbol{I_{pp}}$.

Peak amplitude. The maximum instantaneous value measured from the mean value of a waveform is the peak amplitude. The peak amplitude is E_m or V_m or I_m. The peak amplitude is generally described as the maximum value, hence the maximum voltage has the symbol V_m.

8.3 Average and r.m.s. values of an alternating current

Consider the general case of a current the waveform of which cannot be represented by a simple mathematical expression. For instance, the wave shown below is typical of the current taken by a transformer on no load.

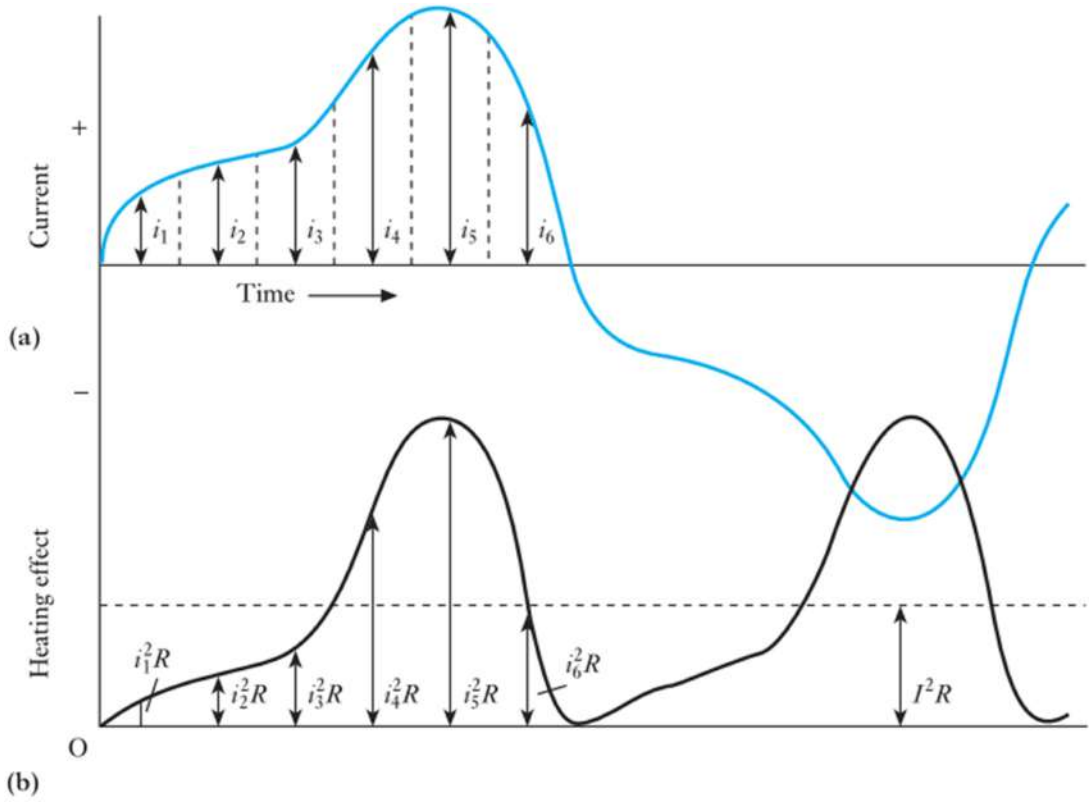


Fig. 10 Average and r.m.s. values

If ***n*** equidistant **mid-ordinates**, i_1, i_2, etc. are taken over either the positive or the negative half-cycle, then *average* value of current over half a cycle is

$$I_{av} = \frac{i_1 + i_2 + \ldots + i_n}{n}$$

Or, alternatively, **average value of current** is

$$\frac{\text{Area enclosed over half-cycle}}{\text{Length of base over half-cycle}}$$

Note: We talk of half-cycle since over a complete cycle the average value is zero.

The area under the curve is found by approximate methods such as the trapezoidal rule, the mid-ordinate rule or Simpson's rule. Average values are represented by $\mathbf{V_{AV}}$, $\mathbf{I_{AV}}$, etc.

In a.c. work, however, the average value is of comparatively little importance. This is due to the fact that it is the power produced by the electric current that usually matters. Thus, if the current represented in Fig. 9.10(a) is passed through a resistor having resistance R ohms, the heating effect of i_1 is i_1^2R, that of i_2 is i_2^2R, etc. as shown in Fig. 9.10(b). The variation of the heating effect during the second half-cycle is exactly the same as that during the first half-cycle.

$$\therefore \quad \text{Average heating effect} = \frac{i_1^2R + i_2^2R + \ldots + i_n^2R}{n}$$

Suppose I to be the value of *direct* current through the same resistance R to produce **a heating effect equal to the average heating effect of the alternating current,** then

$$I^2R = \frac{i_1^2R + i_2^2R + \ldots + i_n^2R}{n}$$

$$\therefore \quad I = \sqrt{\left(\frac{i_1^2 + i_2^2 + \ldots + i_n^2}{n}\right)}$$

= square *root* of the *mean* of the *squares* of the current

= root-mean-square (or r.m.s.) value of the current

Thus, the r.m.s. or effective value of an alternating current is measured in terms of the direct current that produces the same heating effect in the same resistance.

Alternatively, the average heating effect can be expressed as follows:

Average heating effect over half-cycle

$$= \frac{\text{area enclosed by } i^2R \text{ curve over half-cycle}}{\text{length of base}}$$

The r.m.s. value is always greater than the average except for a rectangular wave, in which case the heating effect remains constant so that the average and the r.m.s. values are the same.

Form factor of a wave is

$$\frac{\text{RMS value}}{\text{Average value}}$$

Peak or *crest factor* of a wave is

$$\frac{\text{Peak or maximum value}}{\text{RMS value}}$$

In conclusion, the **effective value (rms)** of an alternating current is that current which will produce the same heating effect as an equivalent direct current. The effective value is called the **root mean square (rms) value** and whenever an alternating quantity is given, it is assumed to be the rms value. For example, the domestic mains supply in Kenya is 240 V and is assumed to mean '240 V rms'. The symbols used for rms values are I, V, E, etc.

8.3.1 Average and r.m.s. values of sinusoidal currents and voltages

Consider the figure below.

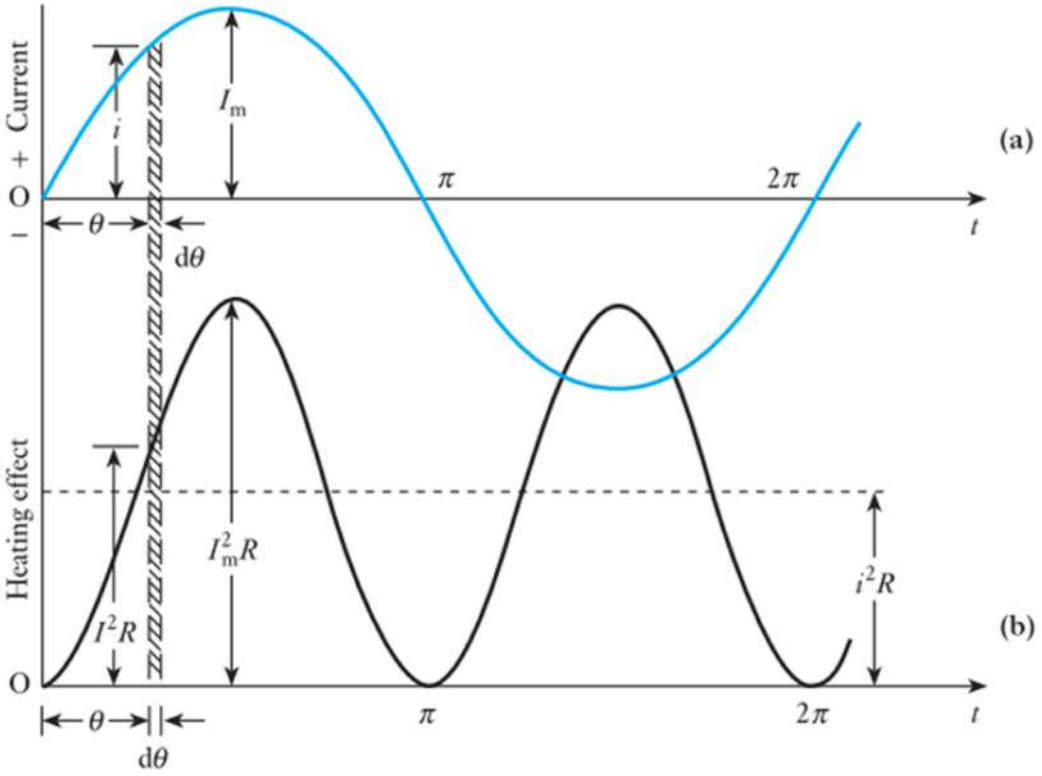


Fig. 9.12 Average and r.m.s. values of a sinusoidal current

If I_m is the maximum value of a current which varies sinusoidally as shown in Fig. 9.12(a), the instantaneous value i is represented by

$$i = I_m \sin\theta$$

where θ is the angle in radians from instant of zero current.

For a very small interval **dθ** radians, the area of the shaded strip is ***i*·dθ** ampere radians. Therefore, **total area** enclose

ed by the current wave over half-cycle is

$$\int_0^{\pi} i \cdot d\theta = I_m \int_0^{\pi} \sin\theta \cdot d\theta = -I_m \left[\cos\theta\right]_0^{\pi}$$

$$= -I_m[-1 - 1] = 2I_m \text{ ampere radians}$$

As seen earlier, average current is given by

$$\frac{\text{Area enclosed over half-cycle}}{\text{Length of base over half-cycle}}$$

Thus, for Fig. 9.12, the average value of current over a half-cycle is

$$\frac{2I_m \text{ [ampere radians]}}{\pi \text{ [radians]}}$$

i.e. $$I_{av} = 0.637 I_m \quad \text{amperes}$$

If the current is passed through a resistor having resistance $\boldsymbol{R}$ ohms, instantaneous heating effect = $\boldsymbol{i^2R}$ watts. The variation of $\boldsymbol{i^2R}$ during a complete cycle is shown in Fig. 9.12(b). During interval $\boldsymbol{d\theta}$ radians, heat generated is $\boldsymbol{i^2R \cdot d\theta}$ watt radians and is represented by the area of the shaded strip. Hence heat generated during the first half-cycle is area enclosed by the i^2R curve and is

$$\int_0^{\pi} i^2R \cdot \mathrm{d}\theta = I_m^2 R \int_0^{\pi} \sin^2\theta \cdot \mathrm{d}\theta$$

$$= \frac{I_m^2 R}{2} \int_0^{\pi} (1 - \cos 2\theta) \cdot \mathrm{d}\theta$$

$$= \frac{I_m^2 R}{2} \left[\theta - \tfrac{1}{2} \sin 2\theta\right]_0^{\pi}$$

$$= \frac{\pi}{2} I_m^2 R \text{ watt radians}$$

As seen previously,

$$\text{Average heating effect over half-cycle} = \frac{\text{area enclosed by } i^2R \text{ curve over half-cycle}}{\text{length of base}}$$

Thus for a sinusoidal waveform, the average heating effect is

$$\frac{(\pi/2)I_m^2 R\ [\text{watt radians}]}{\pi\ [\text{radians}]} = \tfrac{1}{2}I_m^2 R \text{ watts}$$

If ***I*** is the value of direct current through the same resistance to produce the same heating effect

$$I^2R = \tfrac{1}{2}I_m^2 R$$

$$\therefore \qquad I = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

While I is I_{RMS} it is normal practice to omit the RMS subscript, as this is the most common current.

Since the voltage across the resistor is directly proportional to the current, it follows that the relationships derived for currents also apply to voltages. Hence, in general, average value of a sinusoidal current or voltage is

$$0.637 \times \text{maximum value}$$

$$\therefore \qquad I_{av} = 0.637 I_m \qquad [9.15]$$

r.m.s. value of a sinusoidal current or voltage is

$$0.707 \times \text{maximum value}$$

$$\therefore \qquad I = 0.707 I_m \qquad [9.16]$$

From expressions [9.15] and [9.16], form factor of a sine wave is

$$\frac{0.707 \times \text{maximum value}}{0.637 \times \text{maximum value}}$$

$$k_f = 1.11 \qquad [9.17]$$

and peak or crest factor of a sine wave is

$$\frac{\text{maximum value}}{0.707 \times \text{maximum value}}$$

$$\therefore \quad k_p = 1.414 \qquad [9.18]$$

An alternating voltage has the equation $v = 141.4 \sin 377t$; what are the values of:

(a) r.m.s. voltage;
(b) frequency;
(c) the instantaneous voltage when $t = 3$ ms?

The relation is of the form $v = V_m \sin \omega t$ and, by comparison,

(a) $V_m = 141.4 \text{ V} = \sqrt{2}V$

hence $V = \dfrac{141.4}{\sqrt{2}} = \mathbf{100 \text{ V}}$

(b) Also by comparison

$\omega = 377 \text{ rad/s} = 2\pi f$

hence $f = \dfrac{377}{2\pi} = \mathbf{60 \text{ Hz}}$

(c) Finally

$v = 141.4 \sin 377t$

When $t = 3 \times 10^{-3}$ s

$v = 141.4 \sin(377 \times 3 \times 10^{-3}) = 141.4 \sin 1.131$

$= 141.4 \times 0.904 = \mathbf{127.8 \text{ V}}$

Note that, in this example, it was necessary to determine the sine of 1.131 rad, which could be obtained either from suitable tables, or from a calculator. Alternatively, 1.131 rad may be converted into degree measurement, i.e.

$$1.131 \text{ rad} \equiv 1.131 \times \frac{180}{\pi} = 64.8°$$

8.3.2 The equation of a sinusoidal waveform

In Fig 14.7, OA represents a vector that is free to rotate anticlockwise about 0 at an angular velocity of ω rad/s. A rotating vector is known as a **phasor.**

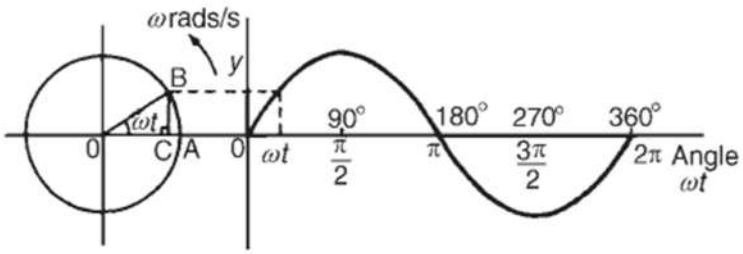


Figure 14.7

After time t seconds the vector OA has turned through an angle ωt. If the line BC is constructed perpendicular to OA as shown, then

$$\sin \omega t = \frac{\text{BC}}{\text{OB}} \quad \text{i.e. BC} = \text{OB} \ \sin \omega t$$

If all such vertical components are projected onto a graph of ***y*** against angle ***ωt*** (in radians), a sine curve results of maximum value OA. **Any quantity which varies sinusoidally can thus be represented as a phasor.**

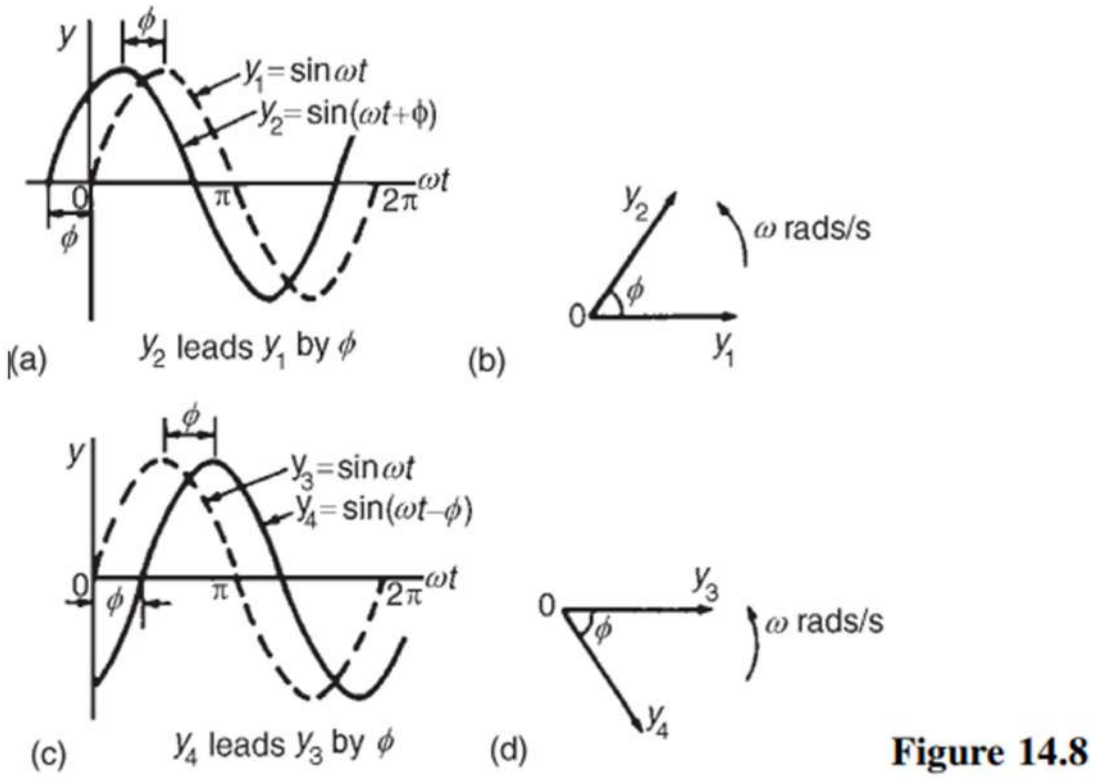


Figure 14.8

A sine curve may not always start at 0°. To show this a periodic function is represented by $y = \sin(\omega t \pm \phi)$, where ϕ is the phase (or angle) difference compared with $y = \sin \omega t$. In Figure 14.8(a), $y_2 = \sin(\omega t + \phi)$ starts ϕ radians earlier than $y_1 = \sin \omega t$ and is thus said to **lead** y_1 by ϕ radians. Phasors y_1 and y_2 are shown in Figure 14.8(b) at the time when $t = 0$.

In Figure 14.8(c), $y_4 = \sin(\omega t - \phi)$ starts ϕ radians later than $y_3 = \sin \omega t$ and is thus said to **lag** y_3 by ϕ radians. Phasors y_3 and y_4 are shown in Figure 14.8(d) at the time when $t = 0$.

Given the general sinusoidal voltage, $\boldsymbol{v = V_m \sin(\omega t \pm \phi)}$, then

(i) Amplitude or maximum value $= V_m$
(ii) Peak to peak value $= 2V_m$
(iii) Angular velocity $= \omega$ rad/s
(iv) Periodic time, $T = 2\pi/\omega$ seconds
(v) Frequency, $f = \omega/2\pi$ Hz (since $\omega = 2\pi$ f)
(vi) ϕ = angle of lag or lead (compared with $v = V_m \sin \omega t$)

If $\boldsymbol{\Phi \neq 0}$, we also say that y_1 and y_2 and are *out of phase*. Otherwise they are said to be *in phase*; they reach their minima and maxima at exactly the same time. We can compare y_1 and y_2 in this manner because they operate at the same frequency; they do not need to have the

same amplitude.

Problem 9. An alternating voltage is given by $v = 282.8 \sin 314t$ volts. Find (a) the rms voltage, (b) the frequency and (c) the instantaneous value of voltage when $t = 4$ ms

(a) The general expression for an alternating voltage is

$$v = V_m \sin(\omega t \pm \phi).$$

Comparing $v = 282.8 \sin 314t$ with this general expression gives the peak voltage as 282.8 V

Hence the rms voltage $= 0.707 \times$ maximum value

$$= 0.707 \times 282.8 = \mathbf{200\ V}$$

(b) Angular velocity, $\omega = 314$ rad/s, i.e. $2\pi f = 314$

Hence frequency, $f = \dfrac{314}{2\pi} = \mathbf{50\ Hz}$

(c) When $t = 4$ ms, $v = 282.8 \sin(314 \times 4 \times 10^{-3})$

$$= 282.8 \sin(1.256) = \mathbf{268.9\ V}$$

(Note that 1.256 radians $= \left[1.256 \times \dfrac{180}{\pi}\right]^{\circ} = 71.96^{\circ} = 71°58'$

Hence $v = 282.8 \sin 71°58' = \mathbf{268.9\ V}$)

9 Sinusoids and Phasors

A sinusoid is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as *alternating current* (*ac*). Such a current reverses at regular time intervals and has alternately positive and negative values. **Circuits driven by sinusoidal current or voltage sources are called *ac circuits*.**

A sinusoidal signal is easy to generate and transmit. It is the form of voltage generated throughout the world and supplied to homes, factories, laboratories, and so on. It is the dominant form of signal in the communications and electric power industries.

9.1 Sinusoids

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t \qquad (9.1)$$

where

V_m = the *amplitude* of the sinusoid
ω = the *angular frequency* in radians/s
ωt = the *argument* of the sinusoid

The sinusoid is shown in Fig. 9.1(a) as a function of its argument and in Fig. 9.1(b) as a function of time.

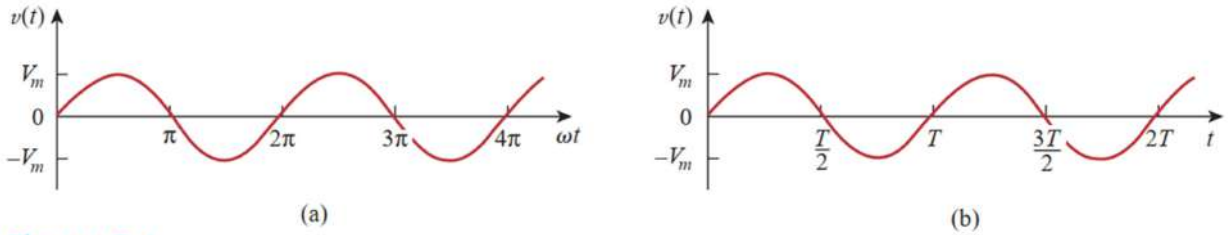


Figure 9.1
A sketch of $V_m \sin \omega t$: (a) as a function of ωt, (b) as a function of t.

It is evident that the sinusoid repeats itself every T seconds; thus, T is called the *period* of the sinusoid. From the two plots in Fig. 9.1, we observe that $\omega T = 2\pi$,

$$\boxed{T = \frac{2\pi}{\omega}} \qquad (9.2)$$

The fact that $v(t)$ repeats itself every T seconds is shown by replacing t by $t + T$ in Eq. (9.1). We get

$$\begin{aligned} v(t + T) &= V_m \sin\omega(t + T) = V_m \sin\omega\left(t + \frac{2\pi}{\omega}\right) \\ &= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t) \end{aligned} \qquad (9.3)$$

Hence,

$$\boxed{v(t + T) = v(t)} \qquad (9.4)$$

that is, v has the same value at $t + T$ as it does at t and $v(t)$ is said to be *periodic*. In general,

A periodic function is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

Now,

$$\boxed{f = \frac{1}{T}} \qquad (9.5)$$

From Eqs. (9.2) and (9.5), it is clear that

$$\omega = 2\pi f \qquad (9.6)$$

While ω is in radians per second (rad/s), f is in hertz (Hz).

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient / convenient to express both as either sine or cosine with positive amplitudes. This is achieved by using the following trigonometric identities:

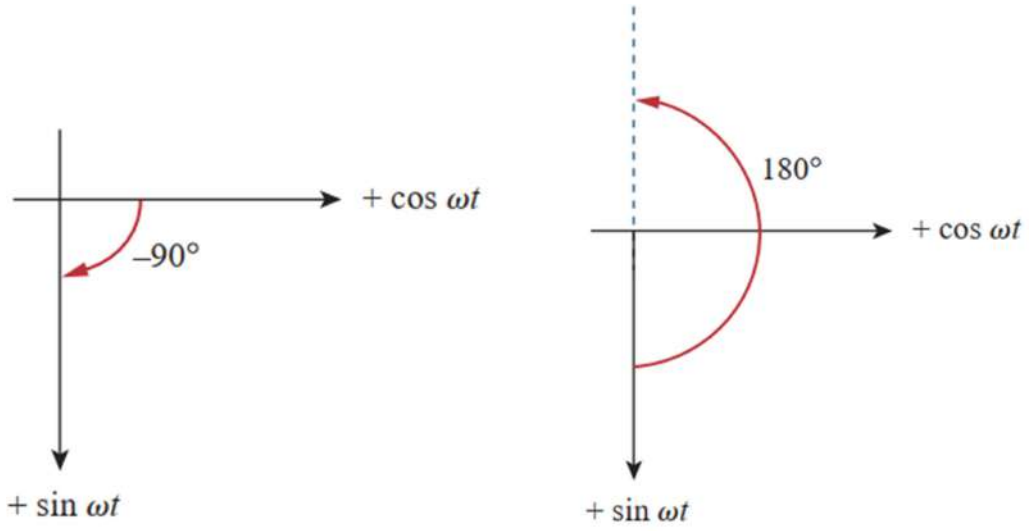
$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned} \qquad (9.9)$$

With these identities, it is easy to show that

$$\begin{aligned}
\sin(\omega t \pm 180^\circ) &= -\sin\omega t \\
\cos(\omega t \pm 180^\circ) &= -\cos\omega t \\
\sin(\omega t \pm 90^\circ) &= \pm\cos\omega t \\
\cos(\omega t \pm 90^\circ) &= \mp\sin\omega t
\end{aligned} \qquad \textbf{(9.10)}$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

A graphical approach may be used to relate or compare sinusoids as an alternative to using the trigonometric identities in Eqs. (9.9) and (9.10). Consider the set of axes shown in Fig. 9.3(a). The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates. This graphical technique can be used to relate two sinusoids. For example, we see in Fig. 9.3(a) that subtracting 90° from the argument of $\cos\omega t$ gives $\sin\omega t$, or $\cos(\omega t - 90^\circ) = \sin\omega t$. Similarly, adding 180° to the argument of $\sin\omega t$ gives $-\sin\omega t$, or $\sin(\omega t + 180^\circ) = -\sin\omega t$, as shown in Fig. 9.3(b).



(a) (b)

Figure 9.3
A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90^\circ) = \sin\omega t$, (b) $\sin(\omega t + 180^\circ) = -\sin\omega t$.

The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form. To add $A\cos\omega t$ and $B\sin\omega t$, we note that A is the magnitude of $\cos\omega t$ while B is the magnitude of $\sin\omega t$ as shown in Fig.9.4 (a).

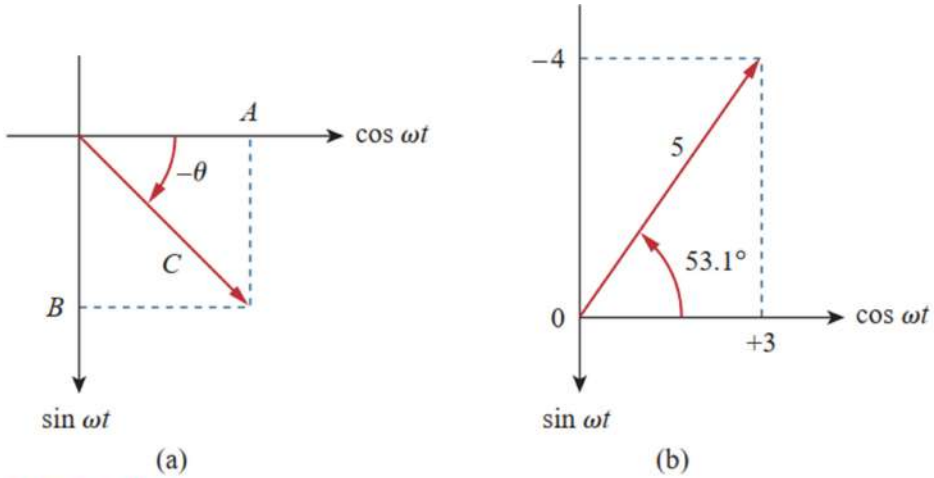


Figure 9.4
(a) Adding $A\cos\omega t$ and $B\sin\omega t$, (b) adding $3\cos\omega t$ and $-4\sin\omega t$.

The magnitude and argument of the resultant sinusoid in cosine form is readily obtained from the triangle. Thus,

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \theta) \qquad \textbf{(9.11)}$$

where

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1}\frac{B}{A} \qquad \textbf{(9.12)}$$

For example, we may add $3\cos\omega t$ and $-4\sin\omega t$ as shown in Fig. 9.4(b) and obtain

$$3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1°) \qquad \textbf{(9.13)}$$

Compared with the trigonometric identities in Eqs. (9.9) and (9.10), the graphical approach eliminates memorization. However, we must not confuse the sine and cosine axes with the axes for complex numbers. Something else to note in Figs. 9.3 and 9.4 is that although the natural tendency is to have the vertical axis point up, the positive direction of the sine function is down in the present case.

Example:

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50°)$ and $v_2 = 12\sin(\omega t - 10°)$. State which sinusoid is leading.

Solution:
Let us calculate the phase in three ways. The first two methods use trigonometric identities, while the third method uses the graphical approach.

■ **METHOD 3** We may regard v_1 as simply $-10\cos\omega t$ with a phase shift of $+50°$. Hence, v_1 is as shown in Fig. 9.5. Similarly, v_2 is $12\sin\omega t$ with a phase shift of $-10°$, as shown in Fig. 9.5. It is easy to see from Fig. 9.5 that v_2 leads v_1 by $30°$, that is, $90° - 50° - 10°$.

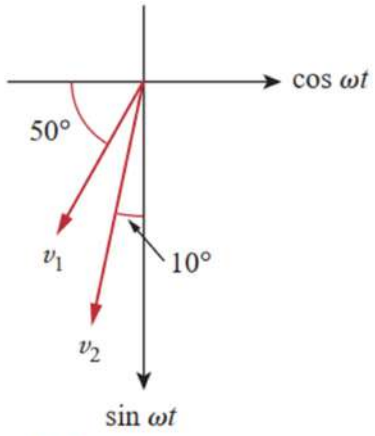


Figure 9.5

Practice Problem 9.2 Find the phase angle between

$$i_1 = -4\sin(377t + 25°) \quad \text{and} \quad i_2 = 5\cos(377t - 40°)$$

Does i_1 lead or lag i_2?

Answer: $155°$, i_1 leads i_2.

9.2 Phasors

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number z can be written in rectangular form as

$$z = x + jy \tag{9.14a}$$

The complex number z can also be written in polar or exponential form as

$$z = r\angle\phi = re^{j\phi} \tag{9.14b}$$

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$$\begin{aligned} z &= x + jy && \text{Rectangular form} \\ z &= r\angle\phi && \text{Polar form} \\ z &= re^{j\phi} && \text{Exponential form} \end{aligned} \tag{9.15}$$

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x} \tag{9.16a}$$

On the other hand, if we know r and ϕ, we can obtain x and y as

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{9.16b}$$

Thus, z may be written as

$$\boxed{z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)} \tag{9.17}$$

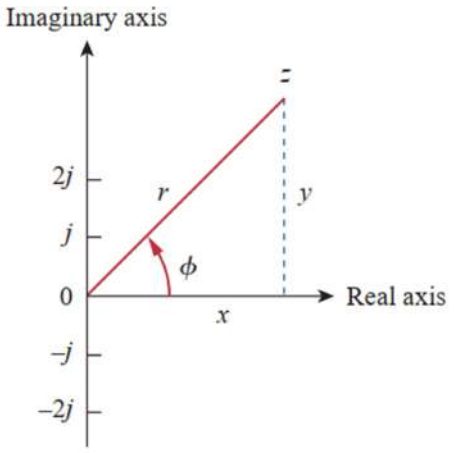


Figure 9.6
Representation of a complex number $z = x + jy = r\angle\phi$.

Import notes on complex numbers

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \qquad \textbf{(9.18a)}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \qquad \textbf{(9.18b)}$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2 \qquad \textbf{(9.18c)}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_2 \qquad \textbf{(9.18d)}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r} \angle -\phi \qquad \textbf{(9.18e)}$$

Square Root:

$$\sqrt{z} = \sqrt{r} \angle \phi/2 \qquad \textbf{(9.18f)}$$

Complex Conjugate:

$$z^* = x - jy = r\angle -\phi = re^{-j\phi} \qquad \textbf{(9.18g)}$$

Note that from Eq. (9.18e),

$$\frac{1}{j} = -j \qquad \textbf{(9.18h)}$$

The idea of phasor representation is based on Euler's identity. In general,

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \tag{9.19}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \text{Re}(e^{j\phi}) \tag{9.20a}$$

$$\sin\phi = \text{Im}(e^{j\phi}) \tag{9.20b}$$

where Re and Im stand for the *real part of* and the *imaginary part of*. Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we use Eq. (9.20a) to express $v(t)$ as

$$v(t) = V_m \cos(\omega t + \phi) = \text{Re}(V_m e^{j(\omega t + \phi)}) \tag{9.21}$$

or

$$v(t) = \text{Re}(V_m e^{j\phi} e^{j\omega t}) \tag{9.22}$$

Thus,

$$v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) \tag{9.23}$$

where

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle\phi \tag{9.24}$$

V is thus the *phasor representation* of the sinusoid as we said earlier. In other words, **a phasor is a complex representation of the magnitude and phase of a sinusoid.**

Since a phasor has magnitude and phase ("direction"), it behaves as a vector and is printed in boldface. For example, phasors

$$\mathbf{V} = V_m \angle\phi \text{ and } \mathbf{I} = I_m \angle -\theta$$

are graphically represented in Fig. 9.8. Such a graphical representation of phasors is known as **a *phasor diagram.***

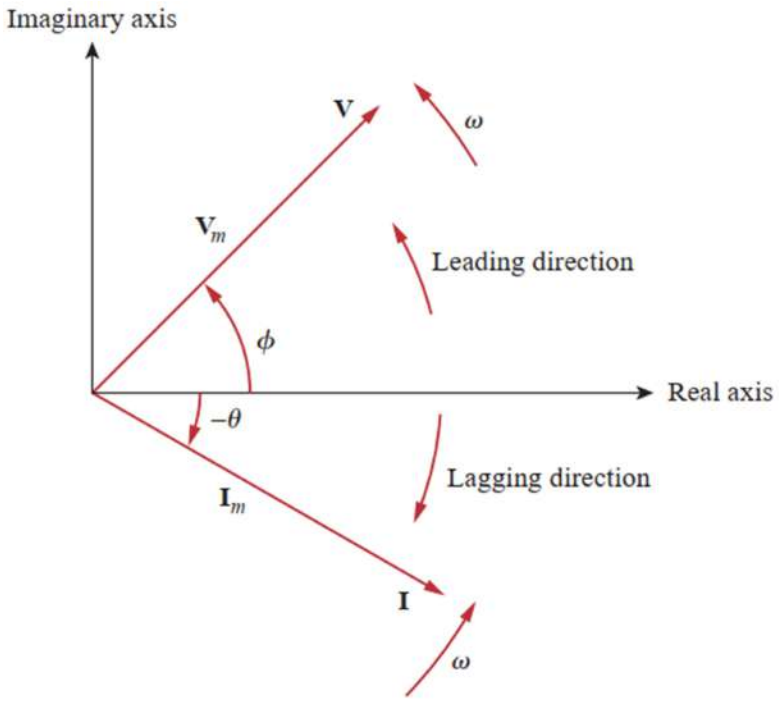


Figure 9.8
A phasor diagram showing $\mathbf{V} = V_m \angle \phi$ and $\mathbf{I} = I_m \angle -\theta$.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi \tag{9.25}$$

(Time-domain representation) ⇔ (Phasor-domain representation)

Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we obtain the corresponding phasor as $\mathbf{V} = V_m \angle \phi$. Equation (9.25) is also demonstrated in Table 9.1, where the sine function is considered in addition to the cosine function. From Eq. (9.25), we see that to get the phasor representation of a sinusoid, we express it in cosine form and take the magnitude and phase. Given a phasor, we obtain the time domain representation as the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor. The idea of expressing information in alternate domains is fundamental to all areas of engineering.

Equations (9.21) through (9.23) reveal that to get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$, and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$\underbrace{v(t) = V_m \cos(\omega t + \phi)}_{\text{(Time-domain representation)}} \quad \Leftrightarrow \quad \underbrace{\mathbf{V} = V_m \angle \phi}_{\text{(Phasor-domain representation)}} \tag{9.25}$$

TABLE 9.1

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90°$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90°$

Note that in Eq. (9.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the *frequency domain.*

From Eqs. (9.23) and (9.24), $v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$, so that

$$\begin{aligned} \frac{dv}{dt} &= -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90°) \\ &= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90°}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t}) \end{aligned} \tag{9.26}$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\underbrace{\frac{dv}{dt}}_{\text{(Time domain)}} \quad \Leftrightarrow \quad \underbrace{j\omega \mathbf{V}}_{\text{(Phasor domain)}} \tag{9.27}$$

That is, differentiating a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$.

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\underbrace{\int v\,dt}_{\text{(Time domain)}} \quad \Leftrightarrow \quad \underbrace{\frac{\mathbf{V}}{j\omega}}_{\text{(Phasor domain)}} \tag{9.28}$$

That is, integrating a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$.

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time domain* representation, while $\mathbf{V}$ is the *frequency or phasor domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

Finally, we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

Example 9.4

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40°)$ A
(b) $v = -4\sin(30t + 50°)$ V

Solution:

(a) $i = 6\cos(50t - 40°)$ has the phasor

$$\mathbf{I} = 6\angle -40° \text{ A}$$

(b) Since $-\sin A = \cos(A + 90°)$,

$$v = -4\sin(30t + 50°) = 4\cos(30t + 50° + 90°)$$
$$= 4\cos(30t + 140°) \text{ V}$$

The phasor form of v is

$$\mathbf{V} = 4\angle 140° \text{ V}$$

Example 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

Solution:

(a) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ)\text{ A}$$

(b) Since $j = 1\angle 90^\circ$,

$$\mathbf{V} = j8\angle -20^\circ = (1\angle 90^\circ)(8\angle -20^\circ)$$
$$= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ\text{ V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ)\text{ V}$$

Adding sinusoids of the same frequency is equivalent to adding their corresponding phasors.

Example 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ A and $i_2(t) = 5\sin(\omega t - 20^\circ)$ A, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\angle 30^\circ$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I}_2 = 5\angle -110^\circ$$

If we let $i = i_1 + i_2$, then

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 4\angle 30^\circ + 5\angle -110^\circ$$
$$= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698$$
$$= 3.218\angle -56.97^\circ\text{ A}$$

Transforming this to the time domain, we get

$$i(t) = 3.218\cos(\omega t - 56.97^\circ)\text{ A}$$

Of course, we can find $i_1 + i_2$ using Eq. (9.9), but that is the hard way.

9.2.1 Phasor Relationships for Circuit Elements

To obtain phasor relationships for the passive (*R, L, C*) elements, we transform the voltage-current relationship from the time domain to the frequency domain for each element.

We begin with the resistor. If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi) \tag{9.29}$$

The phasor form of this voltage is

$$\mathbf{V} = RI_m \angle \phi \tag{9.30}$$

But the phasor representation of the current is $\mathbf{I} = I_m \angle \phi$. Hence,

$$\mathbf{V} = R\mathbf{I} \tag{9.31}$$

This shows that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain.

We should note from Eq. (9.31) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 9.10.

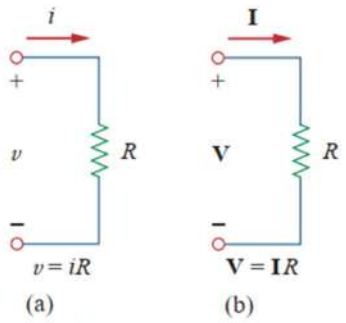


Figure 9.9
Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

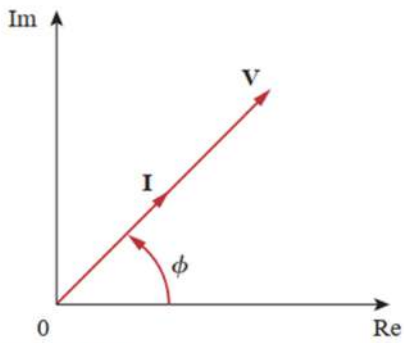


Figure 9.10
Phasor diagram for the resistor.

For the inductor L, assume the current through it is $i = I_m \cos(\omega t + \phi)$. The voltage across the inductor is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \tag{9.32}$$

Recall from Eq. (9.10) that $-\sin A = \cos(A + 90°)$. We can write the voltage as

$$v = \omega L I_m \cos(\omega t + \phi + 90°) \tag{9.33}$$

which transforms to the phasor

$$\mathbf{V} = \omega L I_m e^{j(\phi+90°)} = \omega L I_m e^{j\phi} e^{j90°} = \omega L I_m \angle \phi + 90° \tag{9.34}$$

But $I_m \angle \phi = \mathbf{I}$, and from Eq. (9.19), $e^{j90°} = j$. Thus,

$$\mathbf{V} = j\omega L\mathbf{I} \tag{9.35}$$

This shows that the voltage has a magnitude of ωLI_m and a phase angle of $\Phi + 90°$. The voltage and current are 90° out of phase. **Specifically, the current lags the voltage by 90°**.

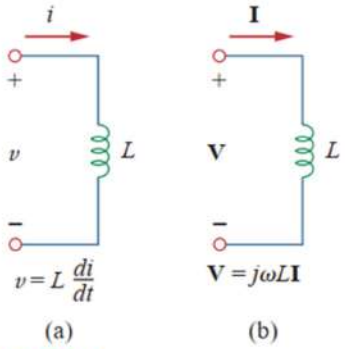


Figure 9.11
Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

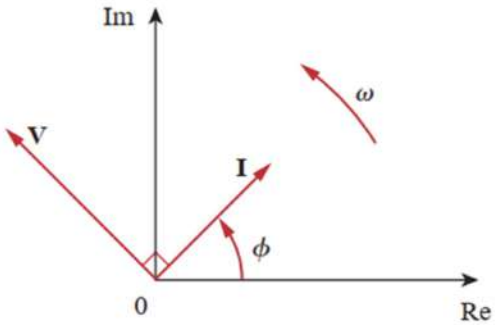


Figure 9.12
Phasor diagram for the inductor; **I** lags **V**.

Although it is equally correct to say that the inductor voltage leads the current by 90°, ***convention gives the current phase relative to the voltage****.*

For the capacitor C, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$. The current through the capacitor is

$$i = C\frac{dv}{dt} \tag{9.36}$$

By following the same steps as we took for the inductor or by applying Eq. (9.27) on Eq. (9.36), we obtain

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.37}$$

This shows that the current and voltage are 90º out of phase. To be specific, the current leads the voltage by 90º.

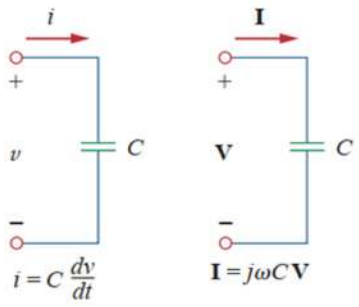


Figure 9.13
Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

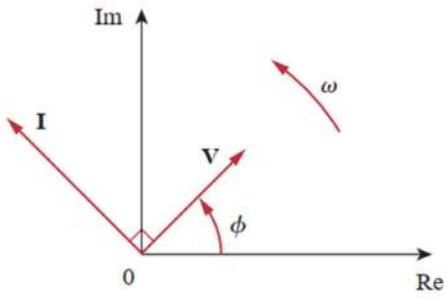


Figure 9.14
Phasor diagram for the capacitor; **I** leads **V**.

TABLE 9.2

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

Example 9.8

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\angle 45°$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\angle 45°}{j60 \times 0.1} = \frac{12\angle 45°}{6\angle 90°} = 2\angle -45° \text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°) \text{ A}$$

Practice Problem 9.8

If voltage $v = 10\cos(100t + 30°)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

Answer: $50\cos(100t + 120°)$ mA.

9.3 Impedance and Admittance

The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

As seen previously,

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.38}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \tag{9.39}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{ZI}} \tag{9.40}$$

The impedance represents the opposition that the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

As a complex quantity, the impedence may be expressed in rectangular form as

$$\mathbf{Z} = R + jX \tag{9.41}$$

where $R = \text{Re}\,\mathbf{Z}$ is the *resistance* and $X = \text{Im}\,\mathbf{Z}$ is the *reactance*. The reactance X may be positive or negative. We say that the impedance is inductive when X is positive or capacitive when X is negative. Thus, impedance $\mathbf{Z} = R + jX$ is said to be *inductive* or lagging since current lags voltage, while impedance $\mathbf{Z} = R - jX$ is capacitive or leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}| \angle \theta \tag{9.42}$$

Comparing Eqs. (9.41) and (9.42), we infer that

$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}| \angle \theta} \tag{9.43}$$

where

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R} \tag{9.44}$$

and

$$R = |\mathbf{Z}| \cos\theta, \qquad X = |\mathbf{Z}| \sin\theta \tag{9.45}$$

The admittance **Y** is the reciprocal of impedance, measured in siemens (S).

The admittance of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$\boxed{\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}} \tag{9.46}$$

As a complex quantity, we may write **Y** as

$$\mathbf{Y} = G + jB \tag{9.47}$$

where G is called the ***conductance*** and B is called the ***susceptance***. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (9.41) and (9.47),

$$G + jB = \frac{1}{R + jX} \tag{9.48}$$

By rationalization,

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} \tag{9.49}$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2} \tag{9.50}$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if $X = 0$, then $G = 1/R$.

TABLE 9.3

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

Example 9.9

Solution:
From the voltage source $10\cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

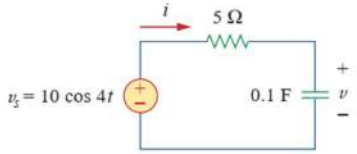


Figure 9.16
For Example 9.9.

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \qquad (9.9.1)$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$
$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V} \qquad (9.9.2)$$

Converting $\mathbf{I}$ and $\mathbf{V}$ in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789\cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47\cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

9.3.1 Kirchhoff's Laws in the Frequency Domain

Kirchhoff's laws holds for phasors.

For KVL, let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then

$$v_1 + v_2 + \cdots + v_n = 0 \qquad (9.51)$$

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0 \qquad (9.52)$$

This can be written as

$$\text{Re}(V_{m1}e^{j\theta_1}e^{j\omega t}) + \text{Re}(V_{m2}e^{j\theta_2}e^{j\omega t}) + \cdots + \text{Re}(V_{mn}e^{j\theta_n}e^{j\omega t}) = 0$$

or

$$\text{Re}[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n})e^{j\omega t}] = 0 \qquad (9.53)$$

If we let $\mathbf{V}_k = V_{mk}e^{j\theta_k}$, then

$$\text{Re}[(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n)e^{j\omega t}] = 0 \qquad (9.54)$$

Since $e^{j\omega t} \neq 0$,

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0 \qquad (9.55)$$

Similarly, If we let $i_1, i_2, \ldots, i_n$ be the current leaving or entering a closed surface in a network at time t, then

$$i_1 + i_2 + \cdots + i_n = 0 \qquad (9.56)$$

If $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ are the phasor forms of the sinusoids $i_1, i_2, \ldots, i_n$, then

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0 \qquad (9.57)$$

Example 9.10

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at $\omega = 50$ rad/s.

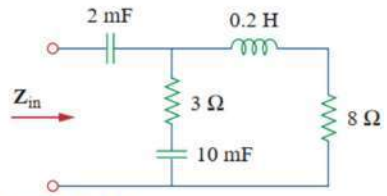


Figure 9.23
For Example 9.10.

Solution:
Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor
$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor
$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_1 + \mathbf{Z}_2 \,\|\, \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Practice Problem 9.10

Determine the input impedance of the circuit in Fig. 9.24 at $\omega = 10$ rad/s.

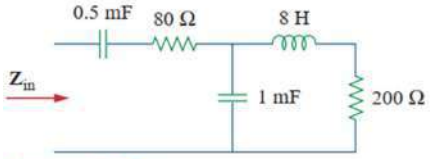


Figure 9.24
For Practice Prob. 9.10.

Answer: $(129.52 - j295)$

Example 9.11

Determine $v_o(t)$ in the circuit of Fig. 9.25.

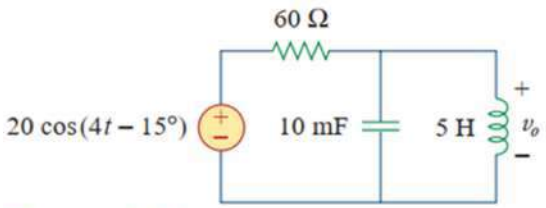


Figure 9.25
For Example 9.11.

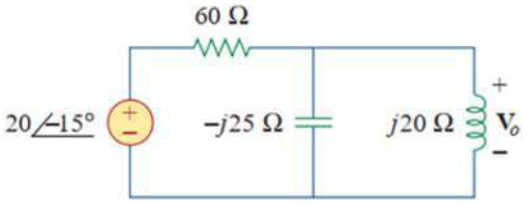


Figure 9.26
The frequency domain equivalent of the circuit in Fig. 9.25.

Solution:

To do the analysis in the frequency domain, we must first transform the time domain circuit in Fig. 9.25 to the phasor domain equivalent in Fig. 9.26. The transformation produces

$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 20\angle -15^\circ\ \text{V}, \qquad \omega = 4$$

$$10\ \text{mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\ \text{H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\angle -15^\circ)$$
$$= (0.8575\angle 30.96^\circ)(20\angle -15^\circ) = 17.15\angle 15.96^\circ\ \text{V}$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ)\ \text{V}$$

Practice Problem 9.11 Calculate v_o in the circuit of Fig. 9.27.

Answer: $v_o(t) = 14.142\cos(10t - 35^\circ)$ V.

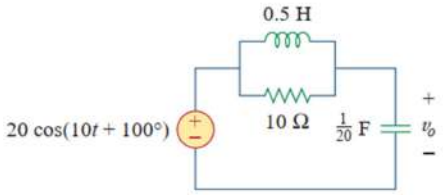


Figure 9.27
For Practice Prob. 9.11.

9.4 Summary

1. A sinusoid is a signal in the form of the sine or cosine function. It has the general form

$$v(t) = V_m\cos(\omega t + \phi)$$

where V_m is the amplitude, $\omega = 2\pi f$ is the angular frequency, $(\omega t + \phi)$ is the argument, and ϕ is the phase.

2. A phasor is a complex quantity that represents both the magnitude and the phase of a sinusoid. Given the sinusoid $v(t) = V_m\cos(\omega t + \phi)$, its phasor $\mathbf{V}$ is

$$\mathbf{V} = V_m\angle\phi$$

3. In ac circuits, voltage and current phasors always have a fixed relation to one another at any moment of time. If $v(t) = V_m \cos(\omega t + \phi_v)$ represents the voltage through an element and $i(t) = I_m \cos(\omega t + \phi_i)$ represents the current through the element, then $\phi_i = \phi_v$ if the element is a resistor, ϕ_i leads ϕ_v by 90° if the element is a capacitor, and ϕ_i lags ϕ_v by 90° if the element is an inductor.
4. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage across it to the phasor current through it:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R(\omega) + jX(\omega)$$

The admittance $\mathbf{Y}$ is the reciprocal of impedance:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = G(\omega) + jB(\omega)$$

Impedances are combined in series or in parallel the same way as resistances in series or parallel; that is, impedances in series add while admittances in parallel add.

5. For a resistor $\mathbf{Z} = R$, for an inductor $\mathbf{Z} = jX = j\omega L$, and for a capacitor $\mathbf{Z} = -jX = 1/j\omega C$.
6. Basic circuit laws (Ohm's and Kirchhoff's) apply to ac circuits in the same manner as they do for dc circuits; that is,

$$\mathbf{V} = \mathbf{ZI}$$
$$\Sigma\, \mathbf{I}_k = 0 \quad \text{(KCL)}$$
$$\Sigma\, \mathbf{V}_k = 0 \quad \text{(KVL)}$$

10 AC Power Analysis

10.1 Instantaneous and Average Power

The **instantaneous power** (in watts) is the power at any instant of time. The *instantaneous power* $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it. Assuming the passive sign convention,

$$p(t) = v(t)i(t) \tag{11.1}$$

Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{11.2a}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{11.2b}$$

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \tag{11.3}$$

We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{11.4}$$

and express Eq. (11.3) as

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i) \tag{11.5}$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω which is twice the angular frequency of the voltage or current.

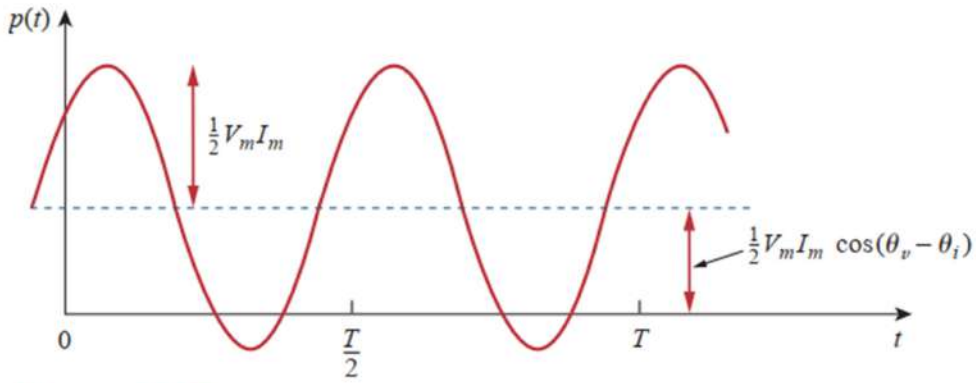


Figure 11.2
The instantaneous power $p(t)$ entering a circuit.

A sketch of $p(t)$ in Eq. (11.5) is shown in Fig. 11.2, where $T = 2\pi/\omega$ is the period of voltage or current. We observe that $p(t)$ is periodic, $p(t) = p(t + T_0)$, and has a period of $T_0 = T/2$, since its frequency is twice that of voltage or current. We also observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle. When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

The instantaneous power changes with time and is therefore difficult to measure. The *average* power is more convenient to measure. In fact, the wattmeter, the instrument for measuring power, responds to average power.

The average power, in watts, is the average of the instantaneous power over one period. Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt \qquad (11.6)$$

Although Eq. (11.6) shows the averaging done over T, we would get the same result if we performed the integration over the actual period of $p(t)$ which is $T_0 = T/2$.

Substituting $p(t)$ in Eq. (11.5) into Eq. (11.6) gives

$$\begin{aligned} P &= \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt \\ &\quad + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt \\ &= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt \\ &\quad + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt \end{aligned} \tag{11.7}$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is cancelled by the area under it during the following negative half-cycle. Thus, the second term in Eq. (11.7) vanishes and the average power becomes

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \tag{11.8}$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and current.

Note that $p(t)$ is time-varying while P does not depend on time. To find the instantaneous power, we must necessarily have $v(t)$ and $i(t)$ in the time domain. But we can find the average power when voltage and current are expressed in the time domain, as in Eq. (11.8), or when they are expressed in the frequency domain. The phasor forms of $v(t)$ and $i(t)$ in Eq. (11.2) are $\mathbf{V} = V_m\angle\theta_v$ and $\mathbf{I} = I_m\angle\theta_i$, respectively. P is calculated using Eq. (11.8) or using phasors $\mathbf{V}$ and $\mathbf{I}$. To use phasors, we notice that

$$\begin{aligned} \frac{1}{2}\mathbf{V}\mathbf{I}^* &= \frac{1}{2}V_m I_m \angle\theta_v - \theta_i \\ &= \frac{1}{2}V_m I_m[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)] \end{aligned} \tag{11.9}$$

We recognize the real part of this expression as the average power P according to Eq. (11.8). Thus,

$$P = \frac{1}{2}\mathrm{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \tag{11.10}$$

$\mathbf{I}^*$ denotes the complex conjugate of $\mathbf{I}$.

Consider two special cases of Eq. (11.10). When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R, and

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R \tag{11.11}$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. Equation (11.11) shows that a purely resistive circuit absorbs power at all times. When $\theta_v - \theta_i = \pm 90°$, we have a purely reactive circuit, and

$$P = \frac{1}{2}V_m I_m \cos 90° = 0 \tag{11.12}$$

showing that a purely reactive circuit absorbs no average power. In summary,

> A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

The average power in Eq. (11.8) can be written in terms of the rms values.

$$\begin{aligned} P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) &= \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i) \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \end{aligned} \tag{11.30}$$

Similarly, the average power absorbed by a resistor R in Eq. (11.11) can be written as

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \tag{11.31}$$

Example 11.1

Given that

$v(t) = 120\cos(377t + 45°)$ V and $i(t) = 10\cos(377t - 10°)$ A

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

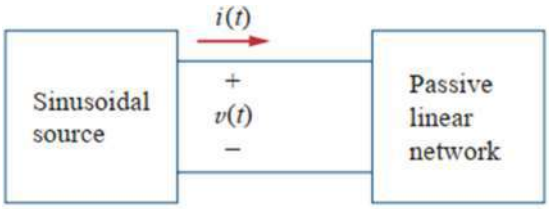


Figure 11.1
Sinusoidal source and passive linear circuit.

Solution:
The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45°)\cos(377t - 10°)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35°) + \cos 55°]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35°)\ \text{W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45° - (-10°)]$$
$$= 600\cos 55° = 344.2\ \text{W}$$

which is the constant part of $p(t)$ above.

Example 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0°$ is applied across it.

Solution:
The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\angle 0°}{30 - j70} = \frac{120\angle 0°}{76.16\angle -66.8°} = 1.576\angle 66.8°\ \text{A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8°) = 37.24\ \text{W}$$

Example 11.3

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

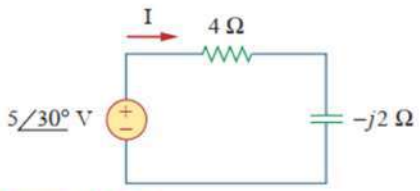


Figure 11.3
For Example 11.3.

Solution:
The current **I** is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I}_R = \mathbf{I} = 1.118\angle 56.57^\circ \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ \text{ V}$$

The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

Example 11.8

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

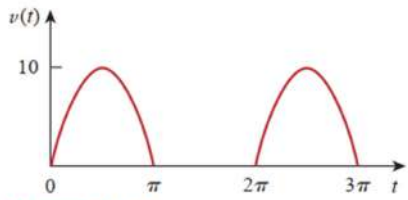


Figure 11.16
For Example 11.8.

Solution:
The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi}(10\sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence,

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi}\frac{100}{2}(1-\cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi}$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5\text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

10.2 Apparent Power and Power Factor

As seen earlier,

$$v(t) = V_m\cos(\omega t + \theta_v) \quad \text{and} \quad i(t) = I_m\cos(\omega t + \theta_i) \qquad \textbf{(11.32)}$$

or, in phasor form, $\mathbf{V} = V_m\angle\theta_v$ and $\mathbf{I} = I_m\angle\theta_i$, the average power is

$$P = \frac{1}{2}V_m I_m\cos(\theta_v - \theta_i) \qquad \textbf{(11.33)}$$

In terms of **rms** values

$$P = V_{\text{rms}}I_{\text{rms}}\cos(\theta_v - \theta_i) = S\cos(\theta_v - \theta_i) \qquad \textbf{(11.34)}$$

We have added a new term to the equation:

$$\boxed{S = V_{\text{rms}}I_{\text{rms}}} \qquad \textbf{(11.35)}$$

The average power is a product of two terms. The product $V_{\text{rms}}I_{\text{rms}}$ is known as the *apparent power S.* The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf). The apparent power (in VA) is the product of the rms values of voltage and current.

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or real power, which is measured in

watts. The power factor is dimensionless, since it is the ratio of the average power to the apparent power. $\theta_v - \theta_i$ (phase difference between voltage and current) is called the *power factor angle*, since it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if **V** is the voltage across the load and **I** is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta_v - \theta_i \qquad (11.37)$$

Alternatively, since

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \angle \theta_v \qquad (11.38a)$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \angle \theta_i \qquad (11.38b)$$

the impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i \qquad (11.39)$$

From Eq. (11.36), the power factor may also be regarded as the ratio of the real (average) power dissipated in the load to the apparent power of the load.

The value of pf ranges between zero and unity. For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf = 1. This implies that the apparent power is equal to the average power. For a purely reactive load, $\theta_v - \theta_i = \pm 90$ and pf = 0. In this case the average power is zero. In between these two extreme cases, pf is said to be *leading* or *lagging*. Leading power factor means that current leads voltage, which implies a capacitive load. Lagging power factor means that current lags voltage, implying an inductive load. Power factor affects the electric bills consumers pay the electric utility companies.

Example 11.9

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10°)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20°)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:
The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20° - 10°) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\angle -20°}{4\angle 10°} = 30\angle -30° = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30°) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C}$$

or

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Practice Problem 11.9

Obtain the power factor and the apparent power of a load whose impedance is $\mathbf{Z} = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 160\cos(377t + 10°)$ V.

Answer: 0.8321 lagging, $177.5\angle 33.69°$ VA.

Example 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

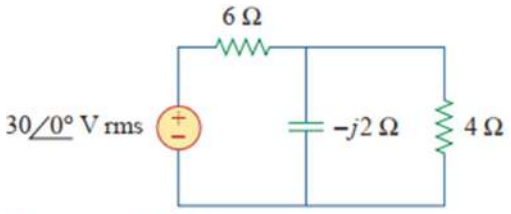


Figure 11.18
For Example 11.10.

Solution:
The total impedance is

$$\mathbf{Z} = 6 + 4 \,\|\, (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\angle -13.24°\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\angle 0^\circ}{7\angle -13.24^\circ} = 4.286\angle 13.24^\circ \text{ A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}} \text{ pf} = (30)(4.286)0.9734 = 125 \text{ W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125 \text{ W}$$

where R is the resistive part of $\mathbf{Z}$.

Practice Problem 11.10

Calculate the power factor of the entire circuit of Fig. 11.19 as seen by the source. What is the average power supplied by the source?

Answer: 0.936 lagging, 1.062 kW.

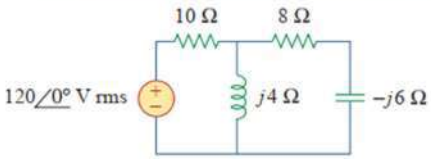


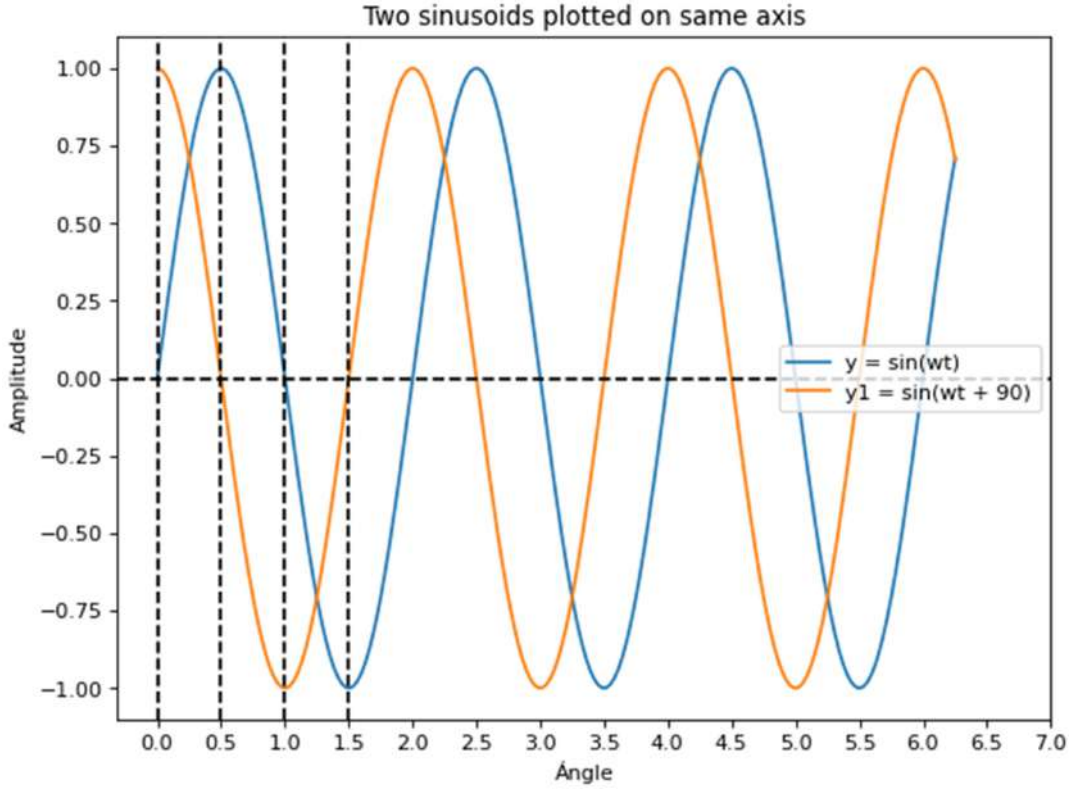
Figure 11.19
For Practice Prob. 11.10.

Note:

1. In a purely resistive load, current and voltage are in phase.
2. In a capacitive load, current leads voltage (by 90^0 in a purely capacitive load).
3. In an inductive load, current lags voltage (by 90^0 in a purely inductive load).
4. In a sinusoidal wave, ***y = A sin (ωt + θ),*** the angle θ giving the phase of the wave (i.e. its shift relative to the wave with θ = 0) is called the phase angle. It represents the fraction of the period that y lags or leads the function, ***A sin (ωt)***. An example is given below in **Python.**

```
import matplotlib.pyplot as plt, numpy as np
t = np.arange(0, 2*np.pi, np.pi/100)
y = np.sin(np.pi*t)
y2 = np.sin(np.pi*t + np.pi/2)
```

```
plt.figure(figsize=(8, 6), dpi=80)
plt.plot(t, y)
plt.plot(t, y2)
plt.axhline(0, color='k', ls = '--')
plt.axvline(0, color='k', ls = '--')
plt.axvline(0.5, color='k', ls = '--')
plt.axvline(1, color='k', ls = '--')
plt.axvline(1.5,color='k', ls = '--')
plt.legend(['y = sin(wt)', 'y1 = sin(wt + 90)'])
plt.xlabel('Ángle')
plt.ylabel('Amplitude')
plt.title('Two sinusoids plotted on same axis')
plt.xticks(np.arange(min(t), max(t)+1, 0.5))
```



The phase difference between the two waveforms (using minima points here) is obtained as: $\theta = (1.5 - 1.0) * \pi * 57.3^0 = 90^0$.

10.3 Complex Power

Complex power is important in power analysis because it contains ***all*** the information pertaining to the power absorbed by a given load.

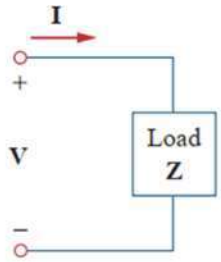


Figure 11.20
The voltage and current phasors associated with a load.

Consider the ac load in Fig. 11.20. Given the phasor form $\mathbf{V} = V_m\angle\theta_v$ and $\mathbf{I} = I_m\angle\theta_i$ of voltage $v(t)$ and current $i(t)$, the *complex power* $\mathbf{S}$ absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* \tag{11.40}$$

assuming the passive sign convention (see Fig. 11.20). In terms of the rms values,

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} \tag{11.41}$$

where

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}}\angle\theta_v \tag{11.42}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}}\angle\theta_i \tag{11.43}$$

Thus we may write Eq. (11.41) as

$$\begin{aligned}\mathbf{S} &= V_{\text{rms}}I_{\text{rms}}\angle\theta_v - \theta_i \\ &= V_{\text{rms}}I_{\text{rms}}\cos(\theta_v - \theta_i) + jV_{\text{rms}}I_{\text{rms}}\sin(\theta_v - \theta_i)\end{aligned} \tag{11.44}$$

We notice from Eq. (11.44) that the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle.

The complex power may be expressed in terms of the load impedance **Z**. From Eq. (11.37), the load impedance **Z** may be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \underline{/\theta_v - \theta_i} \qquad \textbf{(11.45)}$$

Thus, $\mathbf{V}_{\text{rms}} = \mathbf{Z}\mathbf{I}_{\text{rms}}$. Substituting this into Eq. (11.41) gives

$$\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* \qquad \textbf{(11.46)}$$

Since $\mathbf{Z} = R + jX$, Eq. (11.46) becomes

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ \qquad \textbf{(11.47)}$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \qquad \textbf{(11.48)}$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \qquad \textbf{(11.49)}$$

***P* is the average or real power and it depends on the load's resistance *R*. *Q* depends on the load's reactance *X* and is called the *reactive* (or quadrature) power.**

Comparing Eq. (11.44) with Eq. (11.47), we notice that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \qquad \textbf{(11.50)}$$

The real power *P* is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power *Q* is a measure of the energy exchange between the source and the reactive part of the load. The unit of *Q* is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the

reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source. Notice that:

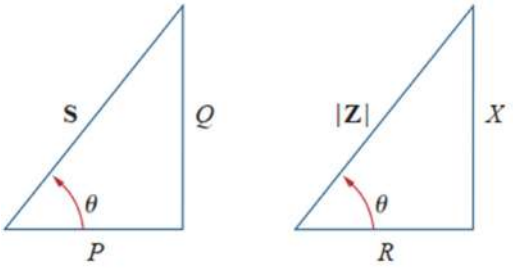
1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus, Complex power (in VA) is the **product of the rms voltage phasor and the complex conjugate of the rms current phasor.** As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

$$\begin{aligned}
\text{Complex Power} = \mathbf{S} &= P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^* \\
&= \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}\angle\theta_v - \theta_i \\
\text{Apparent Power} = S &= |\mathbf{S}| = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}} = \sqrt{P^2 + Q^2} \\
\text{Real Power} = P &= \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\
\text{Reactive Power} = Q &= \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\
\text{Power Factor} &= \frac{P}{S} = \cos(\theta_v - \theta_i)
\end{aligned} \tag{11.51}$$

This shows how the complex power contains *all* the relevant power information in a given load.

It is a standard practice to represent **S**, P, and Q in the form of a triangle, known as the ***power triangle***, shown in Fig. 11.21(a). This is similar to the impedance triangle showing the relationship between **Z**, R, and X, illustrated in Fig. 11.21(b).



(a) (b)

Figure 11.21
(a) Power triangle, (b) impedance triangle.

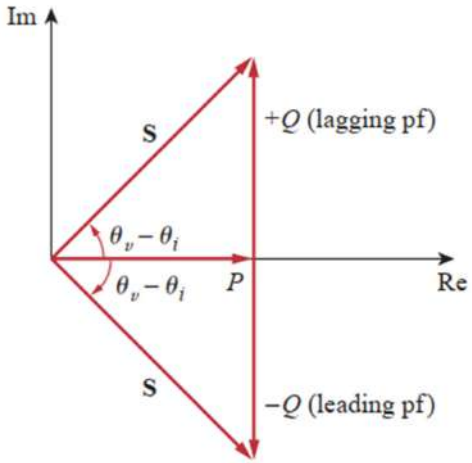


Figure 11.22
Power triangle.

The power triangle has four items—the apparent/complex power, real power, reactive power, and the power factor angle. Given two of these items, the other two can easily be obtained from the triangle. As shown in Fig. 11.22, **when S lies in the first quadrant, we have an inductive load and a lagging pf. When S lies in the fourth quadrant, the load is capacitive and the pf is leading.** It is also possible for the complex power to lie in the second or third quadrant, **but that won't be covered here.**

Example 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10°)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50°)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\angle -10^\circ, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\angle +50^\circ$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = \left(\frac{60}{\sqrt{2}}\angle -10^\circ\right)\left(\frac{1.5}{\sqrt{2}}\angle -50^\circ\right) = 45\angle -60^\circ \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\angle -60^\circ = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10^\circ}{1.5\angle +50^\circ} = 40\angle -60^\circ\ \Omega$$

which is a capacitive impedance.

Practice Problem 11.11

For a load, $\mathbf{V}_{\text{rms}} = 110\angle 85^\circ$ V, $\mathbf{I}_{\text{rms}} = 0.4\angle 15^\circ$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Answer: (a) $44\angle 70^\circ$ VA, 44 VA, (b) 15.05 W, 41.35 VAR, (c) 0.342 lagging, $94.06 + j258.4\ \Omega$.

Example 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that $\text{pf} = \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13^\circ$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}_{rms}^*$, we obtain

$$\mathbf{I}_{rms}^* = \frac{\mathbf{S}}{\mathbf{V}_{rms}} = \frac{10{,}272 + j6204}{120\angle 0°} = 85.6 + j51.7 \text{ A} = 100\angle 31.13° \text{ A}$$

Thus $\mathbf{I}_{rms} = 100\angle -31.13°$ and the peak current is

$$I_m = \sqrt{2}I_{rms} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{rms}}{\mathbf{I}_{rms}} = \frac{120\angle 0°}{100\angle -31.13°} = 1.2\angle 31.13° \ \Omega$$

which is an inductive impedance.

Practice Problem 11.12

A sinusoidal source supplies 20 kVAR reactive power to load $\mathbf{Z} = 250\angle -75° \ \Omega$. Determine: (a) the power factor, (b) the apparent power delivered to the load, and (c) the rms voltage.

Answer: (a) 0.2588 leading, (b) 20.71 kVA, (c) 2.275 kV.

10.4 Power Factor Correction

Most domestic loads (such as washing machines, air conditioners, and refrigerators) and industrial loads (such as induction motors) are inductive and operate at a low lagging power factor. Although the inductive nature of the load cannot be changed, we can increase its power factor.

The process of increasing the power factor without altering the voltage or current to the original load is known as power factor correction.

Since most loads are inductive, as shown in Fig. 11.27(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. 11.27(b). An inductive load is modelled as a series combination of an inductor and a resistor.

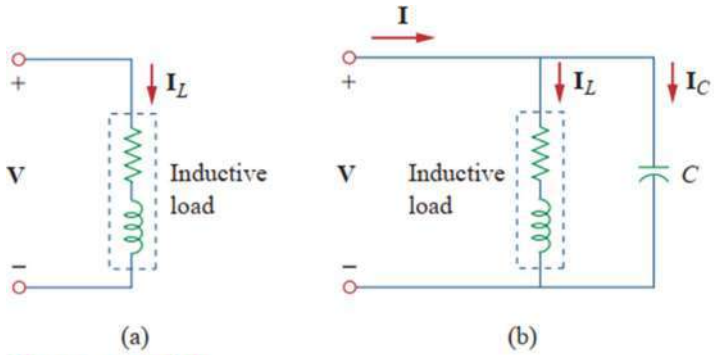


Figure 11.27
Power factor correction: (a) original inductive load, (b) inductive load with improved power factor.

The effect of adding the capacitor can be illustrated using either the power triangle or the phasor diagram of the currents involved. Figure 11.28 shows the latter, where it is assumed that the circuit in Fig. 11.27(a) has a power factor of $\cos\theta_1$, while the one in Fig. 11.27(b) has a power factor of $\cos\theta_2$.

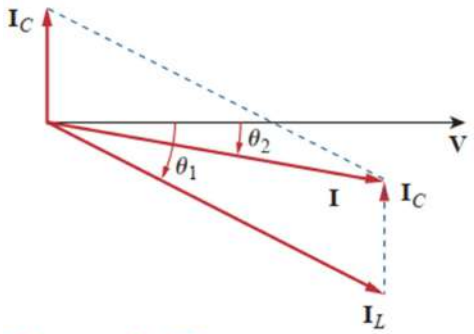


Figure 11.28
Phasor diagram showing the effect of adding a capacitor in parallel with the inductive load.

It is evident from Fig. 11.28 that adding the capacitor has caused the phase angle between the supplied voltage and current to reduce from θ_1 to θ_2 thereby increasing the power factor. We also notice from the magnitudes of the vectors in Fig. 11.28 that with the same supplied voltage, the circuit in Fig. 11.27(a) draws larger current I_L than the current I drawn by the circuit in Fig. 11.27(b). Power companies charge more for larger currents, because they result in increased power losses (by a squared factor, since $P = I_L^2R$). Therefore, it is beneficial to both the power company and the consumer that every effort is made to minimize current level or keep the power factor as close to unity as possible. By choosing a suitable size for the

capacitor, the current can be made to be completely in phase with the voltage, implying unity power factor.

We can look at the power factor correction from another perspective. Consider the power triangle in Fig. 11.29.

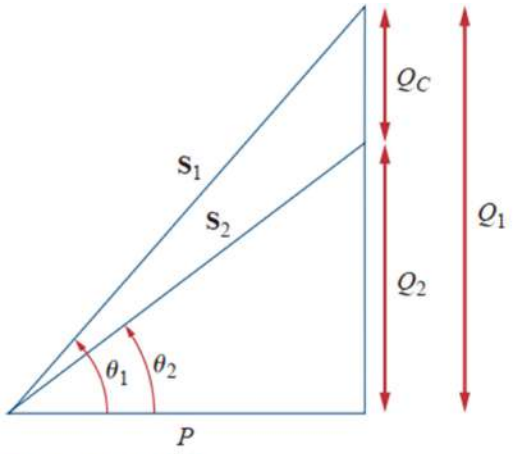


Figure 11.29
Power triangle illustrating power factor correction.

If the original inductive load has apparent power then

$$P = S_1 \cos\theta_1, \qquad Q_1 = S_1 \sin\theta_1 = P \tan\theta_1 \qquad \textbf{(11.57)}$$

If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2 \cos\theta_2$), then the new reactive power is

$$Q_2 = P \tan\theta_2 \qquad \textbf{(11.58)}$$

The reduction in the reactive power is caused by the shunt capacitor; that is,

$$Q_C = Q_1 - Q_2 = P(\tan\theta_1 - \tan\theta_2) \qquad \textbf{(11.59)}$$

But from Eq. (11.46), $Q_C = V_{\text{rms}}^2/X_C = \omega C V_{\text{rms}}^2$. The value of the required shunt capacitance C is determined as

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2} \qquad \textbf{(11.60)}$$

Note that the real power P dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

Although the most common situation in practice is that of an inductive load, it is also possible that the load is capacitive; that is, the load is operating at a leading power factor. In this case, an inductor should be connected across the load for power factor correction. The required shunt inductance L can be calculated from

$$Q_L = \frac{V_{\text{rms}}^2}{X_L} = \frac{V_{\text{rms}}^2}{\omega L} \quad \Rightarrow \quad L = \frac{V_{\text{rms}}^2}{\omega Q_L} \qquad \textbf{(11.61)}$$

where $Q_L = Q_1 - Q_2$, the difference between the new and old reactive powers.

Alternatively, power factor correction may be viewed as the addition of a reactive element (usually a capacitor) in parallel with the load in order to make the power factor closer to unity.

Example 11.15 When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Solution:

If the pf = 0.8, then

$$\cos\theta_1 = 0.8 \quad \Rightarrow \quad \theta_1 = 36.87^\circ$$

where θ_1 is the phase difference between voltage and current. We obtain the apparent power from the real power and the pf as

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000 \text{ VA}$$

The reactive power is

$$Q_1 = S_1 \sin\theta = 5000 \sin 36.87 = 3000 \text{ VAR}$$

When the pf is raised to 0.95,

$$\cos\theta_2 = 0.95 \quad \Rightarrow \quad \theta_2 = 18.19^\circ$$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5 \text{ VA}$$

The new reactive power is

$$Q_2 = S_2 \sin\theta_2 = 1314.4 \text{ VAR}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

and

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$

Note: Capacitors are normally purchased for voltages they expect to see. In this case, the maximum voltage this capacitor will see is about 170 V peak. We would suggest purchasing a capacitor with a voltage rating equal to, say, 200 V.

Practice Problem 11.15

Find the value of parallel capacitance needed to correct a load of 140 kVAR at 0.85 lagging pf to unity pf. Assume that the load is supplied by a 110-V (rms), 60-Hz line.

Answer: 30.69 mF.

10.4.1 Electricity Consumption Cost

Loads with low power factors are costly to serve because they require large currents, as explained earlier. The ideal situation would be to draw minimum current from a supply so that $S = P$, $Q = 0$, and pf = 1. A load with nonzero Q means that energy flows back and forth between the load and the source, giving rise to additional power losses. In view of this, power companies often encourage their customers to have power factors as close to unity as possible and penalize some customers who do not improve their load power factors.

In many countries, utility companies divide their customers into categories: as residential (domestic), commercial, and industrial, or as small power, medium power, and large power. They have different rate structures for each category. The amount of energy consumed in

units of kilowatthours (kWh) is measured using a kilowatt-hour meter installed at the customer's premises.

10.5 Summary

1. The instantaneous power absorbed by an element is the product of the element's terminal voltage and the current through the element:

$$p = vi.$$

2. Average or real power P (in watts) is the average of instantaneous power p:

$$P = \frac{1}{T}\int_0^T p\,dt$$

If $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, then $V_{\text{rms}} = V_m/\sqrt{2}$, $I_{\text{rms}} = I_m/\sqrt{2}$, and

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

Inductors and capacitors absorb no average power, while the average power absorbed by a resistor is $(1/2)I_m^2 R = I_{\text{rms}}^2 R$.

3. Maximum average power is transferred to a load when the load impedance is the complex conjugate of the Thevenin impedance as seen from the load terminals, $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^*$.

4. The effective value of a periodic signal $x(t)$ is its root-mean-square (rms) value.

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

For a sinusoid, the effective or rms value is its amplitude divided by $\sqrt{2}$.

5. The power factor is the cosine of the phase difference between voltage and current:

$$\text{pf} = \cos(\theta_v - \theta_i)$$

It is also the cosine of the angle of the load impedance or the ratio of real power to apparent power. The pf is lagging if the current lags voltage (inductive load) and is leading when the current leads voltage (capacitive load).

6. Apparent power S (in VA) is the product of the rms values of voltage and current:

$$S = V_{\text{rms}} I_{\text{rms}}$$

It is also given by $S = |\mathbf{S}| = \sqrt{P^2 + Q^2}$, where P is the real power and Q is reactive power.

7. Reactive power (in VAR) is:

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

8. Complex power $\mathbf{S}$ (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. It is also the complex sum of real power P and reactive power Q.

$$\mathbf{S} = \mathbf{V}_{\text{rms}} \mathbf{I}^*_{\text{rms}} = V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} = P + jQ$$

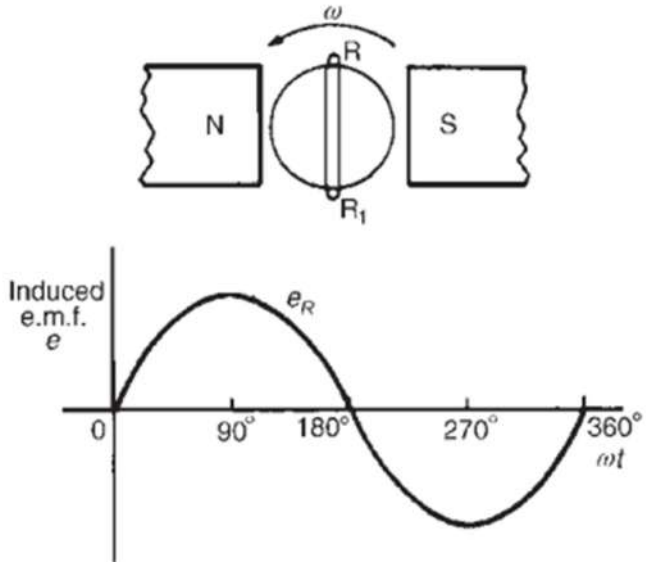
Also,

$$\mathbf{S} = I^2_{\text{rms}} \mathbf{Z} = \frac{V^2_{\text{rms}}}{\mathbf{Z}^*}$$

9. The total complex power in a network is the sum of the complex powers of the individual components. Total real power and reactive power are also, respectively, the sums of the individual real powers and the reactive powers, but the total apparent power is not calculated by the process.
10. Power factor correction is necessary for economic reasons; it is the process of improving the power factor of a load by reducing the overall reactive power.
11. The wattmeter is the instrument for measuring the average power. Energy consumed is measured with a kilowatt-hour meter.

11 Three-Phase Circuits

The voltage induced by a single coil when rotated in a uniform magnetic field is shown below and is known as a **single-phase voltage**.



Most consumers are fed by means of a single-phase a.c. supply. Two wires are used, one called the live conductor (usually coloured red) and the other is called the neutral conductor (usually coloured black). The neutral is usually connected via protective gear to earth, the earth wire being coloured green. The standard voltage for a single-phase a.c. supply is 240 V. The majority of single-phase supplies are obtained by connection to a three-phase supply. A single-phase ac power system consists of a generator connected through a pair of wires (a transmission line) to a load. Figure 12.1(a) depicts a single-phase two wire system, where is the rms magnitude of the source voltage and is the phase.

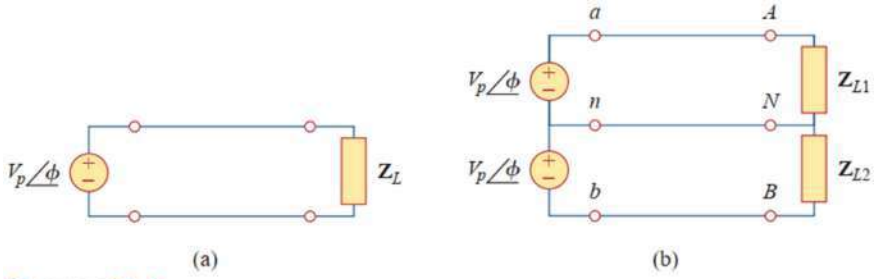


Figure 12.1
Single-phase systems: (a) two-wire type, (b) three-wire type.

Fig. 12.1(b) depicts a single-phase three wire system. It contains two identical sources (equal magnitude and the same phase) that are connected to two loads by two outer wires and the neutral. Such a system allows the connection of both 120-V and 240-V appliances.

Single phase systems are inefficient for transmission and in induction motors.

Circuits or systems in which the ac sources operate at the same frequency but different phases are known as *polyphase.* Figure 12.2 shows a two-phase three-wire system, and Fig. 12.3 shows a three-phase four wire system.

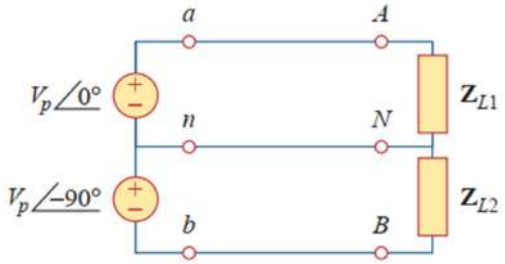


Figure 12.2
Two-phase three-wire system.

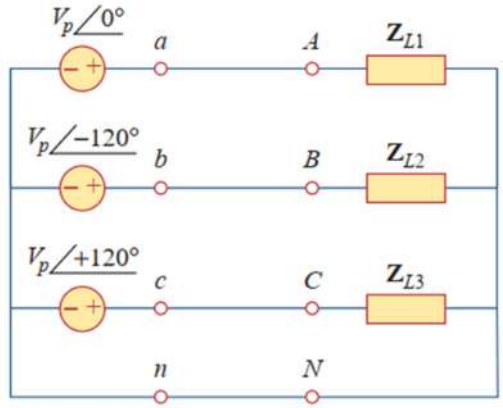


Figure 12.3
Three-phase four-wire system.

As distinct from a single-phase system, a two-phase system is produced by a generator consisting of two coils placed perpendicular to each other so that the voltage generated by one lags the other by 90º. By the same token, a three-phase system is produced by a generator consisting of three sources having the same amplitude and frequency but out of phase with each other by 120º. Three-phase system is by far the most prevalent and most economical poly-phase system.

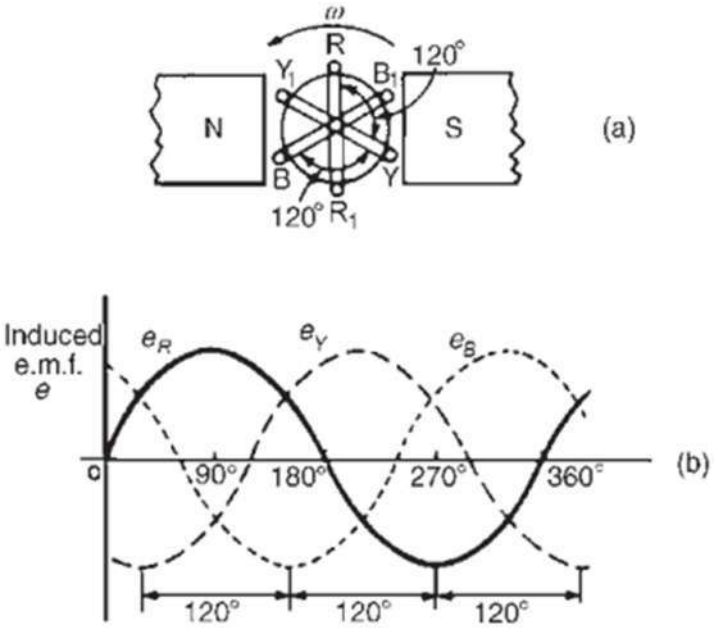
Three-phase systems are important for at least three reasons. First, nearly all electric power is generated and distributed in three-phase. When one-phase or two-phase inputs are required, they are taken from the three-phase system rather than generated independently. Even when more than three phases are needed—such as in the aluminium industry, where 48 phases are required for melting purposes—they can be provided by manipulating the three phases supplied. Second, the instantaneous power in a three-phase system can be constant (not

pulsating). This results in uniform power transmission and less vibration of three-phase machines. Third, for the same amount of power, the three-phase system is more economical than the single phase. The amount of wire required for a three-phase system is less than that required for an equivalent single-phase system.

11.1 Generation of three-phase e.m.f.s

A three-phase supply is generated when three coils are placed 120° apart and the whole rotated in a uniform magnetic field as shown in Figure (a). The result is three independent supplies of equal voltages which are each displaced by 120° from each other as shown in Figure (b).

The convention adopted to identify each of the phase voltages is: R-red, Y-yellow, and B-blue. The **phase-sequence** is given by the sequence in which the conductors pass the point initially taken by the red conductor, e.g. **R, Y, B**.



In the figure above, the conductors are rotating while the magnetic poles are stationary. In most practical cases, it is the other way round as in shown in Fig. 12.4 below.

Three-phase voltages are often produced with a three-phase ac generator (or alternator) whose cross-sectional view is shown in Fig. 12.4.

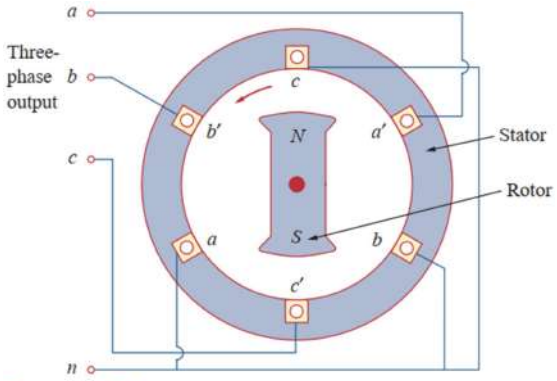


Figure 12.4
A three-phase generator.

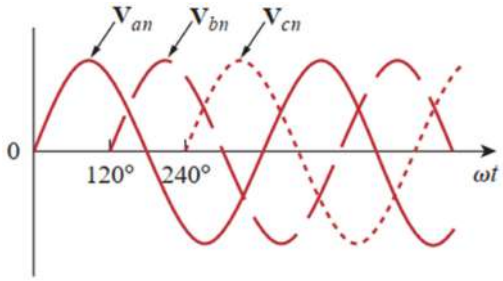


Figure 12.5
The generated voltages are 120° apart from each other.

The generator basically consists of a rotating magnet (called the *rotor*) surrounded by a stationary winding (called the *stator*). Three separate windings or coils with terminals *a-a'*, *b-b'*, and *c-c'* are physically placed apart around the stator. Terminals *a* and *a',* for example, stand for one of the ends of coils going into and the other end coming out of the page. As the rotor rotates, its magnetic field "cuts" the flux from the three coils and induces voltages in the coils. Because the coils are placed 120º apart, the induced voltages in the coils are equal in magnitude but out of phase by 120º (Fig. 12.5). Since each coil can be regarded as a single-phase generator by itself, the three-phase generator can supply power to both single-phase and three-phase loads.

A typical three-phase system consists of three voltage sources connected to loads by three or four wires (or transmission lines). A three-phase system is equivalent to three single-phase circuits. The voltage sources can be either wye-connected as shown in Fig. 12.6(a) or delta-connected as in Fig. 12.6(b).

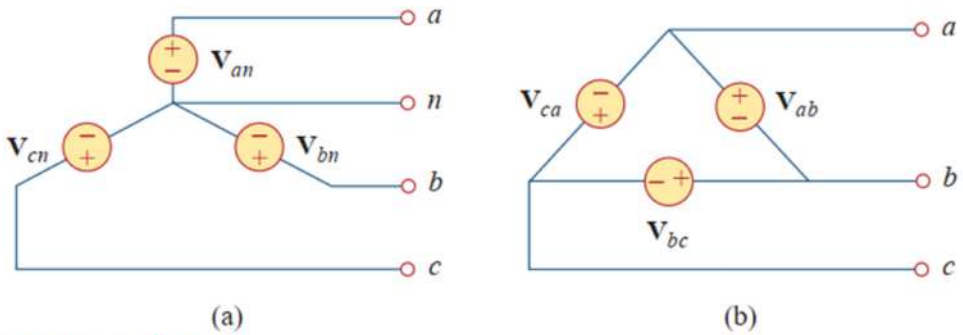


Figure 12.6
Three-phase voltage sources: (a) Y-connected source, (b) Δ-connected source.

Let us consider the wye-connected voltages in Fig. 12.6(a). The voltages and are respectively between lines *a*, *b*, and *c*, and the neutral line *n*. These voltages are called *phase voltages*. If the voltage sources have the same amplitude and frequency and are out of phase with each other by the voltages are said to be *balanced*. This implies that

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0 \qquad \textbf{(12.1)}$$

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| \qquad \textbf{(12.2)}$$

Thus,

Balanced phase voltages are equal in magnitude and are out of phase with each other by 120°.

Since the three-phase voltages are 120º out of phase with each other, there are two possible combinations:

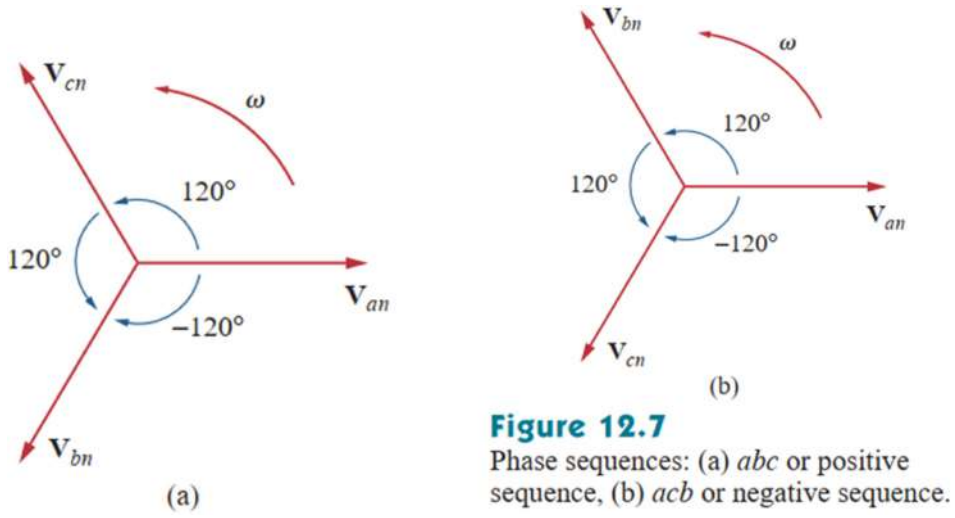


Figure 12.7
Phase sequences: (a) *abc* or positive sequence, (b) *acb* or negative sequence.

One possibility is shown in Fig. 12.7(a) and expressed mathematically as

$$\begin{aligned} \mathbf{V}_{an} &= V_p \angle 0° \\ \mathbf{V}_{bn} &= V_p \angle -120° \\ \mathbf{V}_{cn} &= V_p \angle -240° = V_p \angle +120° \end{aligned} \qquad \textbf{(12.3)}$$

where V_p is the effective or rms value of the phase voltages. This is known as the *abc sequence* or *positive sequence*. In this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$, which in turn leads $\mathbf{V}_{cn}$. This sequence is produced when the rotor in Fig. 12.4 rotates counterclockwise. The other possibility is shown in Fig. 12.7(b) and is given by

$$\begin{aligned} \mathbf{V}_{an} &= V_p\angle 0^\circ \\ \mathbf{V}_{cn} &= V_p\angle -120^\circ \\ \mathbf{V}_{bn} &= V_p\angle -240^\circ = V_p\angle +120^\circ \end{aligned} \qquad \textbf{(12.4)}$$

This is called the *acb sequence* or *negative sequence*. For this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$, which in turn leads $\mathbf{V}_{bn}$. The *acb* sequence is produced when the rotor in Fig. 12.4 rotates in the clockwise direction. It is easy to show that the voltages in Eqs. (12.3) or (12.4) satisfy Eqs. (12.1) and (12.2). For example, from Eq. (12.3),

$$\begin{aligned} \mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} &= V_p\angle 0^\circ + V_p\angle -120^\circ + V_p\angle +120^\circ \\ &= V_p(1.0 - 0.5 - j0.866 - 0.5 + j0.866) \\ &= 0 \end{aligned} \qquad \textbf{(12.5)}$$

The phase sequence is the time order in which the voltages pass through their respective maximum values.

The phase sequence is determined by the order in which the phasors pass through a fixed point in the phase diagram.

In Fig. 12.7(a), as the phasors rotate in the counterclockwise direction with frequency ω, they pass through the horizontal axis in a sequence *abcabca* . . . . Thus, the sequence is *abc* or *bca* or *cab*. Similarly, for the phasors in Fig. 12.7(b), as they rotate in the counterclockwise direction, they pass the horizontal axis in a sequence *acbacba* . . . . This describes the *acb* sequence. The phase sequence is important in three-phase power distribution. It determines the direction of the rotation of a motor connected to the power source, for example.

Like the generator connections, a three-phase load can be either wye-connected or delta-connected, depending on the end application. Figure 12.8(a) shows a wye-connected load, and Fig. 12.8(b) shows a delta-connected load. The neutral line in Fig. 12.8(a) may or may not be there, depending on whether the system is four- or three-wire. (And, of course, a neutral connection is topologically impossible for a delta connection.) A wye- or delta-connected load is said to be *unbalanced* if the phase impedances are not equal in magnitude or phase.

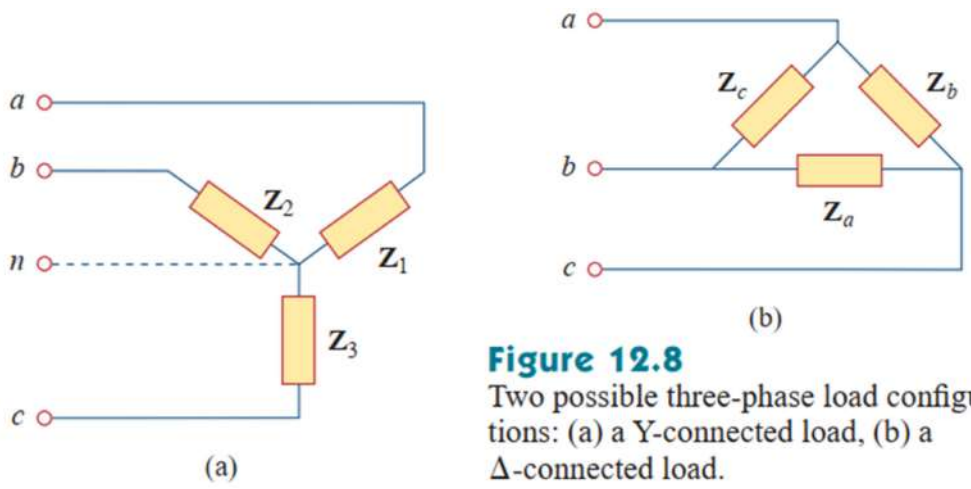


Figure 12.8
Two possible three-phase load configurations: (a) a Y-connected load, (b) a Δ-connected load.

A balanced load is one in which the phase impedances are equal in magnitude and in phase.

For a *balanced* wye-connected load,

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y \qquad (12.6)$$

where $\mathbf{Z}_Y$ is the load impedance per phase. For a *balanced* delta-connected load,

$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta \qquad (12.7)$$

where $\mathbf{Z}_\Delta$ is the load impedance per phase in this case. We recall from Eq. (9.69) that

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \quad \text{or} \quad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta \qquad (12.8)$$

so we know that a wye-connected load can be transformed into a delta-connected load, or vice versa, using Eq. (12.8).

Since both the three-phase source and the three-phase load can be either wye- or delta-connected, we have four possible connections:

- Y-Y connection (i.e., Y-connected source with a Y-connected load).
- Y-Δ connection.
- Δ-Δ connection.
- Δ-Y connection.

It is appropriate to mention here that a balanced delta-connected load is more common than a balanced wye-connected load. This is due to the ease with which loads may be added or removed from each phase of a delta-connected load. This is very difficult with a wye-connected load because the neutral may not be accessible. On the other hand, delta-connected sources are not common in practice because of the circulating current that will result in the delta-mesh if the three-phase voltages are slightly unbalanced.

Example 12.1

Determine the phase sequence of the set of voltages

$$v_{an} = 200\cos(\omega t + 10^\circ)$$
$$v_{bn} = 200\cos(\omega t - 230^\circ), \qquad v_{cn} = 200\cos(\omega t - 110^\circ)$$

Solution:

The voltages can be expressed in phasor form as

$$\mathbf{V}_{an} = 200\angle 10^\circ \text{ V}, \qquad \mathbf{V}_{bn} = 200\angle -230^\circ \text{ V}, \qquad \mathbf{V}_{cn} = 200\angle -110^\circ \text{ V}$$

We notice that $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120° and $\mathbf{V}_{cn}$ in turn leads $\mathbf{V}_{bn}$ by 120°. Hence, we have an *acb* sequence.

Practice Problem 12.1

Given that $\mathbf{V}_{bn} = 110\angle 30^\circ$ V, find $\mathbf{V}_{an}$ and $\mathbf{V}_{cn}$, assuming a positive (*abc*) sequence.

Answer: $110\angle 150^\circ$ V, $110\angle -90^\circ$ V.

11.2 Balanced Wye-Wye Connection

A balanced Y-Y system is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.

Any balanced three-phase system can be reduced to an equivalent Y-Y system. Therefore, analysis of this system should be regarded as the key to solving all balanced three-phase systems.

Consider the balanced four-wire Y-Y system of Fig. 12.9, where a Y-connected load is connected to a Y-connected source.

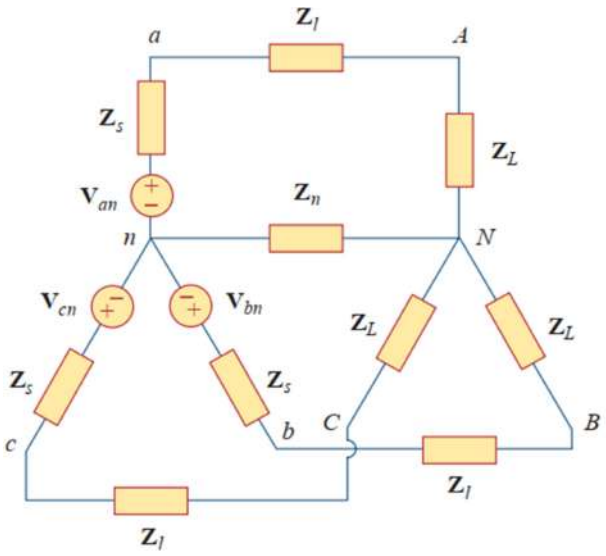


Figure 12.9
A balanced Y-Y system, showing the source, line, and load impedances.

We assume a balanced load so that load impedances are equal. Although the impedance Z_Y is the total load impedance per phase, it may also be regarded as the sum of the source impedance Z_s, line impedance Z_l and load impedance Z_L for each phase, since these impedances are in series. As illustrated in Fig. 12.9, Z_s denotes the internal impedance of the phase winding of the generator; Z_l is the impedance of the line joining a phase of the source with a phase of the load; Z_L is the impedance of each phase of the load; and Z_n is the impedance of the neutral line.

Thus, in general

$$\mathbf{Z}_Y = \mathbf{Z}_s + \mathbf{Z}_\ell + \mathbf{Z}_L \qquad (12.9)$$

$\mathbf{Z}_s$ and $\mathbf{Z}_\ell$ are often very small compared with $\mathbf{Z}_L$, so one can assume that $\mathbf{Z}_Y = \mathbf{Z}_L$ if no source or line impedance is given. In any event, by lumping the impedances together, the Y-Y system in Fig. 12.9 can be simplified to that shown in Fig. 12.10.

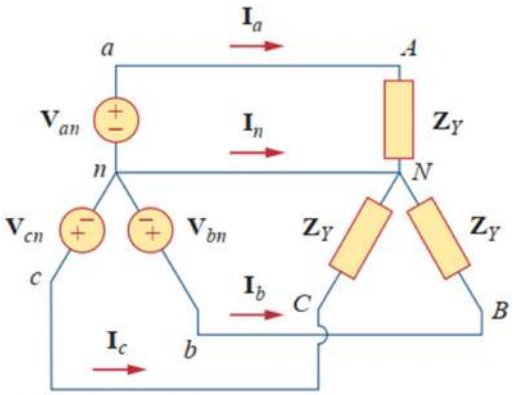


Figure 12.10
Balanced Y-Y connection.

An easier way of analysing a balanced Y-Y system is to do so on a "per phase" basis. We look at one phase, say phase *a*, and analyse the single-phase equivalent circuit in Fig. 12.12.

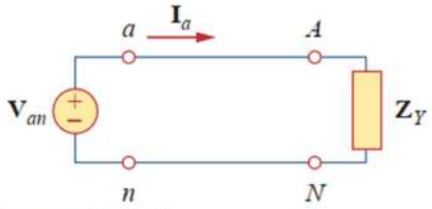


Figure 12.12
A single-phase equivalent circuit.

The single-phase analysis yields the line current $\mathbf{I}_a$ as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} \qquad (12.18)$$

From $\mathbf{I}_a$ we use the phase sequence to obtain other line currents. Thus, as long as the system is balanced, we need only analyse one phase. We may do this even if the neutral line is absent, as in the three-wire system.

Calculate the line currents in the three-wire Y-Y system of Fig. 12.13. **Example 12.2**

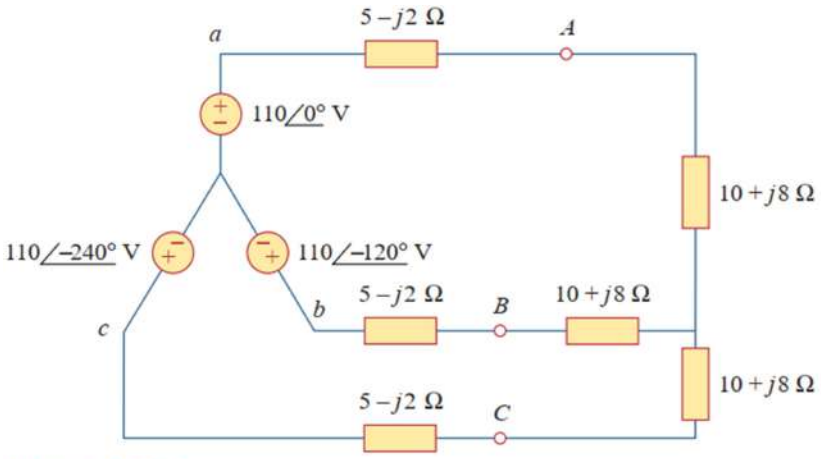


Figure 12.13
Three-wire Y-Y system; for Example 12.2.

Solution:

The three-phase circuit in Fig. 12.13 is balanced; we may replace it with its single-phase equivalent circuit such as in Fig. 12.12. We obtain $\mathbf{I}_a$ from the single-phase analysis as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$$

where $\mathbf{Z}_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\angle 21.8°$. Hence,

$$\mathbf{I}_a = \frac{110\angle 0°}{16.155\angle 21.8°} = 6.81\angle -21.8° \text{ A}$$

Since the source voltages in Fig. 12.13 are in positive sequence, the line currents are also in positive sequence:

$$\mathbf{I}_b = \mathbf{I}_a\angle -120° = 6.81\angle -141.8° \text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle -240° = 6.81\angle -261.8° \text{ A} = 6.81\angle 98.2° \text{ A}$$